GENERAL NONLINEAR THEORY OF SANDWICH SHELLS

By

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LIST OF SYMBOLS

${}_{0}a^{i}$	components of acceleration vector
$\vec{a}_{i}$; $\vec{a}^{i}$	base vectors in a Euclidean Space of normal coordinates
$a_{\alpha\beta}$	first fundamental form of shell middle surface
A	area of shell middle surface
${}_{0}b^{i}$	components of body force per unit mass
$b_{\alpha\beta}$; b_{β}^{α}	second fundamental form of shell middle surface
${}_{n}B^{\alpha\beta\delta\lambda}$	isothermal stiffnesses
C^{i}	effective external couple resultants measured per unit area of the shell middle surface
${}_{0}C$	edge curve of shell middle surface
$C^{ijk\ell}$	elastic coefficients
$'d^{i}$; $''d^{i}$	acceleration resultants defined in (106)
$'\tilde{e}^{i}$; $''\tilde{e}^{i}$	prescribed external edge forces on facings
$\vec{e}_{i}$	base vector in cartesian coordinates
E_{ijmn}	elastic coefficients
f^{i}	acceleration forces
F^{i}	body forces
g^{i}	effective external edge moments defined in (106)

$\vec{g}_i$; $\vec{G}_i$	base vectors in a Euclidean Space
g_{ij} ; G_{ij}	metric tensors
$2\bar{h}$, $2\,'h$, $2\,''h$	thickness of core and facings, respectively
δI_i	volume integral defined by (93a)-(93c)
δJ	surface integral defined by (93d)
$\tilde{\ell}^i$	prescribed edge moments
m^i	acceleration moment resultants
M^i	body moment resultants
$'m^{\alpha\beta}$, $\bar{m}^{\alpha\beta}$, $''m^{\alpha\beta}$	moment resultants due to $'S^{\alpha\beta}$, $\bar{S}^{\alpha\beta}$, and $''S^{\alpha\beta}$, respectively
$_0n_\alpha$, $_0n_3$	components of normal vector in E-3 space
$'n^{\alpha\beta}$, $\bar{n}^{\alpha\beta}$, $''n^{\alpha\beta}$	stress resultants due to $'S^{\alpha\beta}$, $\bar{S}^{\alpha\beta}$, and $''S^{\alpha\beta}$, respectively
p^i	effective external loads measured per unit area of shell middle surface
$'q^\alpha$, $\bar{q}^\alpha$, $''q^\alpha$	stress resultants due to $'S^{\alpha 3}$, $\bar{S}^{\alpha 3}$, and $''S^{\alpha 3}$, respectively
$\vec{r}$, $\vec{r}^*$; $\vec{R}$	position vectors
s	arc length of a curve on the shell middle surface
S	area of shell surface
$\tilde{S}^i$	prescribed edge forces
s^{ij}	stress tensor
$'t^\alpha$, $\bar{t}^\alpha$, $''t^\alpha$	moment resultants due to $'S^{\alpha 3}$, $\bar{S}^{\alpha 3}$, and $''S^{\alpha 3}$, respectively

T	temperature field
${}_nT^{ij}$	thermal stress and couple resultants
$'u_i, \bar{u}_i, {}^*u_i$	components of displacement vectors
$'v_i, \bar{v}_i, {}''v_i$	components of displacement vectors defined by (100)
$'V_i, \bar{V}_i, {}^*V_i$	component of displacement vectors in E-3 space
z_i	cartesian coordinates
α_{ij}	thermal coefficients
γ_{ij}	strain tensor
$\Gamma^i_{jk}, \Gamma^{*i}_{jk}$	Christoffel symbols
δ	variation symbol
δ_{mn}	Kronecker delta
θ^i	convected coordinates
$'\psi_i, \bar{\Psi}_i, {}^*\Psi_i$	strain measures defined by (97)
Φ	strain energy function
ρ_o	density
μ^{*}_{β}, μ	expressions defined by (65) and (66) respectively
$(\mu^{-1})^{\alpha}_{\beta}$	inverse of μ^{α}_{β} defined in (69)
${}_o\nu_\beta$	component of normal vector to normal surface
Σ	strain energy per unit area of shell middle surface

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GENERAL NONLINEAR THEORY OF SANDWICH SHELLS

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A general nonlinear theory of sandwich shells to the full extent of the nonlinear strain-displacement relations

$$\gamma_{ij} = \frac{1}{2}(V_i|_j + V_j|_i + V^r|_i V_r|_j)$$

has been obtained by means of the modified Hellinger-Reissner variational principle of three-dimensional elasticity. The fundamental equations are in tensor notation and in terms of the undeformed state. By this same technique and by making certain simplifying assumptions in the strain-displacement relations, partially nonlinear theories for sandwich shells are also obtained. These approximations are based on the assumption of small strain but large deflections, thin facings, and soft cores, respectively. These intermediate theories are more suitable for application.

The intermediate and the linearized form of the various sets of equations derived herein coincide with known theories.

CHAPTER I

INTRODUCTION

A sandwich structure is formed by two thin facings of a strong material between which a thick layer of very light-weight and comparatively weak material is sandwiched. The advantage of this kind on construction is the large moment of inertia of the section provided by spacing far apart the load-carrying facings. Accordingly, the applications of sandwich structures in various areas, especially by the aerospace industry, have increased in recent years. As a result, more research work concerning this kind of structure has become desirable. A series of extensive bibliographies (1), (2)* and comprehensive reviews on the analysis of sandwich structures by Habip (3), (4) have appeared recently.

In the early work of Reissner (5), (6), (7) and Wang (8), the simplest model consisting of two facings acting as membranes and a core resisting transverse shear and normal stresses has been employed for deriving the governing equations of sandwich structures. In recent years, it appears that changes in technology and a concern for the optimization of structural elements

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*Underlined numbers in parentheses refer to the Bibliography at the end of this dissertation.

subjected to thermal as well as mechanical loads have brought about several studies of a new type of sandwich construction with strong cores. The theory then takes the flexural rigidity as well as transverse normal deformation of the core into account while including, as usual, the flexural rigidities of the upper and lower facings about their own middle surfaces. Grigoliuk and Chulkov (9) have presented a paper on this subject for the case of small deflection theory of sandwich shells. They consider the core as a three-dimensional body and assume that the displacements can be expressed approximately as a linear function of the transverse coordinate. Wu (10) generalized this to a large-deflection theory of orthotropic sandwich shallow shells. In the latest works of Ebcioglu (11), (12), the nonlinear field equations for the sandwich plates and shells have been obtained by means of the Hamilton principle. For plates, the nonlinear strain-displacement relations are used to the full extent, but partially nonlinear strain-displacement relations are used to the full extent, but partially nonlinear strain-displacement relations only are employed for shells. The present dissertation is inspired by Habip's works (13), (14). The fundamental equations of the theory of plates and shells have been obtained by Habip from the three-dimensional theory of nonlinear elasticity by integration across the thickness of the undeformed plate and by means of a modified version of the Hellinger-Reissner variational theorem of three-dimensional continuum dynamics, respectively, for the case when the "shifted" components of displacement can be assumed to vary linearly through the thickness of

the shell.

The present study is an attempt at obtaining a general nonlinear theory of sandwich shells to the full extent of the general nonlinear strain-displacement relations (39) in tensor notation and in terms of the undeformed state by means of the modified Hellinger-Reissner variational principle of three-dimensional elasticity. The method employed here is similar to that used by Habip but the order of variation and "shifting" is slightly different. In this work, we introduce the geodesic normal coordinate system into the variational equation before we carry out the variation. In the general theory, the results are identical regardless of the order of variation and "shifting" but in partially nonlinear theories the results come out quite different. We shall discuss this in more detail in Chapter V.

In Chapter III, the "exact" fundamental equations, in the sense of using the complete general nonlinear strain-displacements relations, are given. No assumptions about the state of deformation have been made except that the displacements vary linearly through the thickness. But these "exact" equations are, for practical purposes, too complicated; so we introduce some simplifications in Chapter IV. These approximations are based on the assumption of small strain but large deflections, thin facings, and a soft core, respectively. By making these simplifications we arrive at several approximate theories suitable for applications. When these theories are compared to some known results, they do agree exactly.

The present work, as in (11), (12), takes into account the effects of transverse shear and normal stresses as well as rotatory inertia, with different material densities and material constants in each layer of the sandwich shell. In addition, each of the three layers is of different thickness, and no *a priori* limitations are imposed upon the displacement functions until Chapter IV. It is also assumed that the facings and the core are anisotropic, having elastic symmetry with respect to the middle plane of the layers. The effect of steady thermal gradients is also included in the stress-strain relations.

CHAPTER II

PRELIMINARIES

In order to formulate a geometrically nonlinear theory of sandwich shells, some fundamental concepts from tensor analysis, surface geometry, and elasticity will be used repeatedly in this study and are reproduced here for convenient reference. More detailed treatment of these subjects may be obtained in references (15), (16), and (17).

2.1 Outline of Tensor Analysis

A. Convected coordinates and base vectors. Let z^i be a set of fixed rectangular cartesian coordinates. The $\vec{e}_i$ are unit base vectors, and the position of a point in space with coordinates z^i can be defined by the position vector $\vec{r}$, where

$$\vec{r} = z^i \vec{e}_i \quad . \tag{1}$$

The range of Latin indices is 1, 2, 3. Repeated indices are to be summed over their range.

The differential of the position vector in (1) is

$$d\vec{r} = \frac{\partial \vec{r}}{\partial z^i} dz^i = dz^i \vec{e}_i \quad . \tag{2}$$

The length of a line element is defined as

$$ds^2 = d\vec{r} \cdot d\vec{r} = \vec{e}_i \cdot \vec{e}_j \, dz^i dz^j = \delta_{ij} \, dz^i dz^j \quad , \tag{3}$$

where δ_{ij} denotes the Kronecker symbol.

Let us now introduce a general convected coordinate system θ^i , defined by the coordinate transformation

$$\theta^i = \theta^i (z^1, z^2, z^3) \quad , \tag{4}$$

or

$$z^i = z^i (\theta^1, \theta^2, \theta^3) \quad , \tag{5}$$

provided that the Jacobian of (4) does not vanish.

If $\vec{r}$ is regarded as a function of θ^i , and using a comma followed by a subscript i to denote partial differentiation with respect to θ^i ,

$$d\vec{r} = \vec{r}_{,i} \, d\theta^i = \vec{g}_i \, d\theta^i \quad , \tag{6}$$

where

$$\vec{g}_i = \vec{r}_{,i} = \frac{\partial \vec{r}}{\partial \theta^i} \tag{7}$$

are the base vectors of the convected coordinates.

By the chain rule of partial differentiation, the differential $d\theta^i$ can be expressed as

$$d\theta^i = \frac{\partial \theta^i}{\partial z^j} dz^j \tag{8}$$

Substitution of this expression into Equation (6) yields

$$d\vec{r} = \vec{g}_i \frac{\partial \theta^i}{\partial z^j} dz^j \tag{9}$$

which, by comparison with (2), leads to

$$\vec{e}_j = \frac{\partial \theta^i}{\partial z^j} \vec{g}_i \quad , \tag{10a}$$

$$\vec{e}_j \frac{\partial z^j}{\partial \theta^i} = \vec{g}_i \quad . \tag{10b}$$

From the definition of the law of covariant transformation we know that $\vec{g}_i$ are covariant base vectors for the θ^i coordinate system.

By using (3), the length of a line element is

$$ds^2 = d\vec{r} \cdot d\vec{r} = \vec{g}_i \cdot \vec{g}_j \, d\theta^i d\theta^j = g_{ij} \, d\theta^i d\theta^j \quad , \tag{11}$$

where

$$g_{ij} = \vec{g}_i \cdot \vec{g}_j \tag{12}$$

is called the covariant metric tensor for the θ^i coordinate system. From Equations (3), (11), and with the help of Equation (8),

$$\delta_{ij} = \frac{\partial \theta^m}{\partial z^i} \frac{\partial \theta^n}{\partial z^j} g_{mn} \quad . \tag{13}$$

B. Reciprocal base vectors. There is no distinction between covariant and contravariant components of the base vectors $\vec{e}_i$ in a Cartesian coordinate system. Thus we can write $\vec{e}^{\,i} = \vec{e}_i$,

and define contravariant base vectors in the θ^i coordinate system by the law of contravariant transformation (17), namely,

$$\vec{g}^{\,i} = \frac{\partial \theta^i}{\partial z^j} \vec{e}^{\,j} \quad . \tag{14}$$

In view of Equation (10b),

$$\vec{g}^{\,i} \cdot \vec{g}_j = \delta^i_j \quad , \tag{15}$$

so that, in general, the covariant and contravariant base vectors in the θ^i coordinate system are orthogonal and reciprocal to each other. By analogy with Equation (12), the contravariant metric tensor is defined by

$$g^{ij} = \vec{g}^{\,i} \cdot \vec{g}^{\,j} \quad . \tag{16}$$

The following useful formulae are derived from Equations (14), (10), and (15).

$$\vec{g}_i = g_{ij} \vec{g}^{\,j} \quad , \tag{17a}$$

$$\vec{g}^{\,i} = g^{ij} \vec{g}_j \quad , \tag{17b}$$

$$g^{ik} g_{jk} = \delta^i_j \quad . \tag{17c}$$

Thus, the metric tensor can be used to raise or lower indices. The volume element for the convected coordinate system θ^i is given by

$$d\tau = \sqrt{g}\, d\theta^1 d\theta^2 d\theta^3 \quad , \tag{18}$$

where

$$g = \left| g_{ij} \right| \quad . \tag{19}$$

C. <u>Derivative of a vector</u>. The covariant base vectors $\vec{g}_i$ are expressed in terms of the position vector $\vec{r}$ by Equation (7). Since $\vec{r}$ is assumed to be a continuous function of θ^i, it follows that

$$\vec{g}_{i,j} = \vec{g}_{j,i} \quad . \tag{20}$$

Differentiating Equation (10a), and using (10b), we obtain

$$\vec{g}_{i,j} = \delta_{mn} \frac{\partial^2 z^m}{\partial \theta^i \partial \theta^j} \frac{\partial z^n}{\partial \theta^r} \vec{g}^r \quad .$$

Introducing the definitions of Christoffel symbols of the first and second kind, respectively,

$$\Gamma_{ijr} = \delta_{mn} \frac{\partial^2 z^m}{\partial \theta^i \partial \theta^j} \frac{\partial z^n}{\partial \theta^r} \quad ,$$

and

$$\Gamma^m_{ij} = g^{mr} \Gamma_{ijr} = \vec{g}^m \cdot \vec{g}_{i,j} \quad , \tag{21}$$

the derivative of base vectors (20) becomes

$$\vec{g}_{i,j} = \vec{g}_{j,i} = \Gamma_{ijr} \vec{g}^r = \Gamma^m_{ij} \vec{g}_m \quad . \tag{22}$$

Now for any vector $\vec{V}$, which is expressed in terms of its components by

$$\vec{V} = V_i \vec{g}^i = V^i \vec{g}_i \quad , \tag{23}$$

the derivative can be written as

$$\vec{V}_{,j} = (V^i \vec{g}_i)_{,j} = V^i{}_{,j} \vec{g}_i + V^i \vec{g}_{i,j} \quad . \tag{24}$$

By using Equation (22),

$$\vec{V}_{,j} = V^m|_j \vec{g}_m \quad , \tag{25}$$

where

$$V^m|_j = V^m{}_{,j} + \Gamma^m_{ij} V^i \tag{26}$$

is the covariant derivative of the vector V^m .

Similarly,

$$\vec{V}_{,j} = V_m|_j \vec{g}^m \tag{27}$$

where

$$V_i|_j = V_{i,j} - \Gamma^m_{ij} V_m \tag{28}$$

is the covariant derivative of the vector V_i .

2.2 Concepts from Three-dimensional Theory of Elasticity

A. Strain tensor. Let the undeformed body be described by a general right-handed convected coordinate system. θ^i . If the position vector of a point in the undeformed body is denoted by

$$\vec{r} = \vec{r}\,(\theta^1, \theta^2, \theta^3) \quad , \tag{29}$$

then when this body is subjected to load, it will deform into a new configuration

$$\vec{R} = \vec{R}\,(\theta^1, \theta^2, \theta^3) \tag{30}$$

which is the position vector of a point in the deformed body.

The θ^i are material coordinates which are associated with corresponding points in the deformed and undeformed body.

In the undeformed body, the length of a line element is given by

$$ds^2 = g_{ij}\, d\theta^i d\theta^j \qquad . \qquad (31)$$

In the deformed body, the length of the same line element becomes

$$dS^2 = G_{ij}\, d\theta^i d\theta^j \qquad , \qquad (32)$$

where G_{ij} is the metric tensor in the deformed state. The difference in lengths of these line elements is a measure of the deformation experienced by the body as it moves from the undeformed to the deformed position. This deformation is described by the strain tensor defined by

$$dS^2 - ds^2 = 2\gamma_{ij}\, d\theta^i d\theta^j \qquad , \qquad (33)$$

from which it follows that

$$\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}) \qquad . \qquad (34)$$

This shows that γ_{ij} is a symmetric tensor.

The strain tensor can be expressed in terms of the displacement vector $\vec{V}$ by noting that

$$\vec{R} = \vec{r} + \vec{V} \qquad . \qquad (35)$$

By using the formulae given in the previous section, the base

vectors and metric tensor for the convected coordinates θ^i may be defined in both the undeformed and the deformed body. Thus,

$$
\begin{aligned}
\vec{g}_i &= \vec{r}_{,i} \quad , & \vec{G}_i &= \vec{R}_{,i} \quad , \\
g_{ij} &= \vec{g}_i \cdot \vec{g}_j \quad , & G_{ij} &= \vec{G}_i \cdot \vec{G}_j \quad , \\
g^{ir} g_{rj} &= \delta^i_j \quad , & G^{ir} G_{rj} &= \delta^i_j \quad , \\
\vec{g}^i &= g^{ir} \vec{g}_r \quad , & \vec{G}^i &= G^{ir} \vec{G}_r \quad .
\end{aligned}
\tag{36}
$$

In view of Equations (27) and (35), the base vectors $\vec{G}_i$ and metric tensor G_{ij} become

$$\vec{G}_i = (\delta^m_i + V^m|_i) \, \vec{g}_m \quad , \tag{37}$$

$$G_{ij} = (\delta^m_i + V^m|_i)(\delta^n_j + V^n|_j) \, g_{mn} \quad . \tag{38}$$

Then, substituting these into Equation (34),

$$\gamma_{ij} = \frac{1}{2} (V_i|_j + V_j|_i + V^r|_i \, V_r|_j) \quad . \tag{39}$$

B. <u>Stress tensor</u>. The state of stress is here defined in terms of the three-dimensional stress vector, ${}_{\circ}\vec{t}$, per unit area of the undeformed body and associated with a surface in the deformed body whose unit normal in its undeformed position is ${}_{\circ}\vec{n}$, namely (<u>18</u>),

$${}_{\circ}\vec{t} = s^{ij} \, {}_{\circ}n_i \, \vec{G}_j \quad , \tag{40}$$

where

$$ {}_{\circ}\vec{n} = {}_{\circ}n_i \, \vec{g}^{\,i} \quad . $$

By using Equation (37), expression (40) becomes

$$ {}_{\circ}\vec{t} = S^{ir}(\delta^j_r + v^j|_r)\,{}_{\circ}n_i\,\vec{g}_j = t^{ij}\,{}_{\circ}n_i\,\vec{g}_j \quad . \tag{41} $$

The S^{ij} and t^{ij} are contravariant stress tensors measured per unit area of the undeformed body and referred to base vectors in the deformed and undeformed body, respectively.

C. Stress-strain relations. The conventional Duhamel-Neumann stress-strain relations for an isotropic material can be generalized for an anisotropic material as follows (13).

$$ \gamma_{ij} = E_{ijmn}S^{mn} + \alpha_{ij}T \quad . \tag{42} $$

The inverse of (42) is

$$ S^{ij} = C^{ijrs}(\gamma_{rs} - \alpha_{rs}T) \quad . \tag{43} $$

The coefficients E_{ijmn}, C^{ijrs} , are the elastic coefficients of the medium. They depend on the metric tensors and physical properties of the undeformed body. The α_{ij} are the coefficients of thermal expansion. All of these coefficients are also functions of θ^i through the steady temperature field T and satisfy the following symmetry conditions

$$ E_{ijmn} = E_{jimn} = E_{ijnm} = E_{mnij} \quad , $$

$$ C^{ijrs} = C^{jirs} = C^{ijsr} = C^{rsij} \quad , $$

$$ \alpha_{ij} = \alpha_{ji} \quad . $$

For a medium having elastic symmetry with respect to the surface $\theta^3 =$ const., all coefficients containing the index 3 either once or three times vanish (15), i.e.,

$$C^{\alpha\beta 3\gamma} = C^{\alpha 333} = 0 \quad . \tag{44}$$

Equations (43) then reduce to (14)

$$S^{\alpha\beta} = C^{\alpha\beta\delta\nu}(\gamma_{\delta\nu} - \alpha_{\delta\nu}T) + C^{\alpha\beta 33}(\gamma_{33} - \alpha_{33}T) \quad , \tag{45a}$$

$$S^{\alpha 3} = 2\, C^{\alpha 3\beta 3}(\gamma_{\beta 3} - \alpha_{\beta 3}T) \quad , \tag{45b}$$

$$S^{33} = C^{33\delta\nu}(\gamma_{\delta\nu} - \alpha_{\delta\nu}T) + C^{3333}(\gamma_{33} - \alpha_{33}T) \quad . \tag{45c}$$

2.3 Surface Geometry

A shell is a body occupying the space between two surfaces (called faces) a small distance apart. The coordinate system is so chosen that the surface defined by $\theta^3 = 0$ lies midway between the faces. It is called the middle surface. When θ^3 is measured along a line perpendicular to the middle surface, the coordinate system is called normal (17). In a normal coordinate system the position vector $\vec{r}$ for the undeformed body becomes

$$\vec{r} = \vec{r}^{\,*}(\theta^1, \theta^2) + \theta^3 \vec{a}_3 \quad . \tag{46}$$

The vector $\vec{r}^{\,*}$ locates points on the middle surface, and θ^1 and θ^2 are general curvilinear coordinates on this surface. The vector $\vec{a}_3$ is the unit normal to the middle surface, here directed

outward on a surface of constant positive Gaussian curvature.

The covariant base vectors of the middle surface are

$$\vec{a}_\alpha = \vec{r}^*_{,\alpha} \quad . \tag{47}$$

Here and in what follows, we use the convention that Greek indices take on the values 1, 2. The metric tensor of the middle surface is

$$a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta \quad . \tag{48}$$

The contravariant components are defined by

$$a^{\alpha\beta} a_{\beta\lambda} = \delta^\alpha_\lambda \quad . \tag{49}$$

The contravariant base vectors are defined by

$$\vec{a}^\alpha = a^{\alpha\beta} \vec{a}_\beta \quad . \tag{50}$$

The area of an element on the middle surface is

$$dA = \sqrt{a}\; d\theta^1 d\theta^2 \quad , \tag{51}$$

where

$$a = \left| a_{\alpha\beta} \right| = a_{11} a_{22} - (a_{12})^2 \quad . \tag{52}$$

The coefficients of the second quadratic form are defined by

$$b_{\alpha\beta} = - \vec{a}_\alpha \cdot \vec{a}_{3,\beta} = \vec{a}_3 \cdot \vec{a}_{\alpha,\beta} \quad . \tag{53}$$

The mixed components of the second fundamental form are

$$b^\nu_\beta = a^{\nu\lambda} b_{\lambda\beta} \quad , \tag{54}$$

and the contravariant components are

$$b^{\alpha\beta} = a^{\alpha\lambda} b_{\lambda}^{\beta} \quad . \tag{55}$$

The Christoffel symbols for the middle surface can be obtained by evaluating Equation (22) at $\theta^3 = 0$. We have

$$\vec{a}_{\alpha,\beta} = \Gamma^{*\gamma}_{\alpha\beta} \vec{a}_{\gamma} + \Gamma^{*3}_{\alpha\beta} \vec{a}_3 \quad , \tag{56}$$

where an asterisk denotes quantities on the middle surface. Equation (56) may be rewritten as

$$\vec{a}_{\alpha\|\beta} = \Gamma^{*3}_{\alpha\beta} \vec{a}_3 \quad . \tag{57}$$

Here the double vertical line stands for covariant differentiation based on the metric tensor $a_{\alpha\beta}$.

From Equations (53) and (21), when evaluated at $\theta^3 = 0$,

$$\Gamma^{*3}_{\alpha\beta} = \vec{a}_3 \cdot \vec{a}_{\alpha,\beta} = b_{\alpha\beta} \quad , \tag{58}$$

and hence

$$\vec{a}_{\alpha\|\beta} = b_{\alpha\beta} \vec{a}_3 \quad . \tag{59}$$

Multiplying Equation (53) by $\vec{a}^{\alpha}$, we obtain

$$\vec{a}_{3,\beta} = \vec{a}^{3}{}_{,\beta} = \vec{a}^{3}\|_{\beta} = -b_{\alpha\beta} \vec{a}^{\alpha} \quad , \tag{60}$$

or

$$\vec{a}_{3\|\beta} = \vec{a}_{3,\beta} = -b_{\beta}^{\alpha} \vec{a}_{\alpha} \quad , \tag{61}$$

which is Weingarten's formula.

From Equations (59) and (60)

$$b_{\alpha\beta\|\gamma} - b_{\alpha\gamma\|\beta} = 0 \quad , \qquad (62)$$

which are the Mainardi-Codazzi relations.

The covariant base vectors of the space (46) can now be written as (16)

$$\vec{g}_{\alpha} = \vec{a}_{\alpha} + \theta^3 \vec{a}_{3,\alpha} = \mu^{\beta}_{\alpha} \vec{a}_{\beta} \quad , \qquad (63a)$$

$$\vec{g}_3 = \vec{a}_3 \quad , \qquad (63b)$$

and the metric tensor of the space (46) becomes

$$g_{\alpha\beta} = \mu^{\nu}_{\alpha} \mu^{\lambda}_{\beta} \, a_{\nu\lambda} \quad , \qquad (64a)$$

$$g_{\alpha 3} = 0 \quad , \qquad (64b)$$

$$g_{33} = 1 \quad , \qquad (64c)$$

where

$$\mu^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - \theta^3 b^{\alpha}_{\beta} \quad . \qquad (65)$$

From Equations (64) and (65),

$$\mu = \left| \mu^{\alpha}_{\beta} \right| = (g/a)^{1/2} \quad . \qquad (66)$$

Raising the indices α in Equation (63a) leads to

$$\vec{g}^{\delta} = g^{\alpha\delta} \mu^{\beta}_{\alpha} \vec{a}_{\beta} \quad , \qquad (67)$$

or

$$\vec{g}^{\delta} = g^{\alpha\delta} \mu^{\beta}_{\alpha} a_{\beta\gamma} \vec{a}^{\gamma} = (\mu^{-1})^{\delta}_{\gamma} \vec{a}^{\gamma} \quad , \tag{68}$$

where

$$(\mu^{-1})^{\alpha}_{\beta} = g^{\lambda\alpha} \mu^{\gamma}_{\lambda} a_{\gamma\beta} \tag{69}$$

is the inverse of μ^{α}_{β} (16).

2.4 The Relationship Between Space and Surface Quantities in Normal Coordinates

Differentiating Equations (63) with respect to θ^3 ,

$$\vec{g}_{\alpha,\beta} = \mu^{\gamma}_{\alpha} \vec{a}_{\gamma,\beta} + \mu^{\gamma}_{\alpha,\beta} \vec{a}_{\gamma} \quad . \tag{70}$$

From Equation (21),

$$\vec{g}_{\alpha,\beta} = \Gamma^{\gamma}_{\alpha\beta} \vec{g}_{\gamma} + \Gamma^{3}_{\alpha\beta} \vec{g}_{3} \quad . \tag{71}$$

Substituting Equation (56) into (70), and combining with (71),

$$\Gamma^{\gamma}_{\alpha\beta} \vec{g}_{\gamma} + \Gamma^{3}_{\alpha\beta} \vec{g}_{3} = \mu^{\delta}_{\alpha} (\Gamma^{*\delta}_{\alpha\beta} \vec{a}_{\delta} + \Gamma^{*3}_{\delta\beta} \mu^{\delta}_{\alpha} \vec{a}_{3}) + \mu^{\gamma}_{\alpha,\beta} \vec{a}_{\gamma} \quad . \tag{72}$$

Eliminating $\vec{g}_{\alpha}$ and $\vec{g}_{3}$ from Equations (63), Equation (72) yields

$$(\mu^{\delta}_{\alpha\|\beta} + \Gamma^{*\lambda}_{\alpha\beta} \mu^{\delta}_{\lambda} - \Gamma^{\gamma}_{\alpha\beta} \mu^{\delta}_{\gamma}) \vec{a}_{\delta} + (\Gamma^{*3}_{\delta\beta} \mu^{\delta}_{\alpha} - \Gamma^{3}_{\alpha\beta}) \vec{a}_{3} = 0 \tag{73}$$

where

$$\mu^{\gamma}_{\alpha\|\beta} = \mu^{\gamma}_{\alpha,\beta} + \Gamma^{*\gamma}_{\alpha\beta} \mu^{\delta}_{\alpha} - \Gamma^{*\lambda}_{\alpha\beta} \mu^{\gamma}_{\lambda} \tag{74}$$

Equating the coefficients of $\vec{a}_{\delta}$ and $\vec{a}_{3}$ to zero, respectively,

$$\Gamma^{\delta}_{\alpha\beta} = (\mu^{-1})^{\delta}_{\gamma} (\mu^{\gamma}_{\alpha\|\beta} + \Gamma^{*\gamma}_{\alpha\beta}) \quad , \tag{75}$$

$$\Gamma^{3}_{\alpha\beta} = \mu^{\delta}_{\alpha} \Gamma^{*3}_{\delta\beta} \quad . \tag{76}$$

These are the relations between space and surface Christoffel symbols. Equations (75) and (76) have been derived by Naghdi (16) but in a much more lengthy and indirect manner.

From Equations (23),

$$\vec{V} = V_\alpha \vec{g}^\alpha + V_3 \vec{g}^3$$

$$= V^\alpha \vec{g}_\alpha + V^3 \vec{g}_3 \quad . \tag{77}$$

Referring $\vec{V}$ to the space (46) with base vectors $\vec{a}_\alpha$, $\vec{a}_3$,

$$\vec{V} = V^*_\alpha \vec{a}^\alpha + V^*_3 \vec{a}^3 = V^{*\alpha} \vec{a}_\alpha + V^{*3} \vec{a}_3 \quad . \tag{78}$$

Comparing Equations (77) and (78), with Equations (63) in mind,

$$V_\alpha = \mu^\nu_\alpha V^*_\nu \quad , \qquad V^\alpha = (\mu^{-1})^\alpha_\nu V^{*\nu} \quad ,$$

$$V^*_\alpha = (\mu^{-1})^\nu_\alpha V_\nu \quad , \qquad V^{*\alpha} = \mu^\alpha_\nu V^\nu \quad , \tag{79}$$

$$V_3 = V^3 = V^*_3 = V^{*3} \quad .$$

Equations (79) are the relations between space and surface vectors (16). V^{*3} is a surface invariant. The μ^α_β and its inverse act as "shifters" in the space of normal coordinates.

From Equation (28),

$$V_{\alpha|\beta} = V_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta} V_\lambda - \Gamma^3_{\alpha\beta} V_3 \quad , \tag{80}$$

which, with the help of Equations (65), (76), and (79), reduces to

$$V_{\alpha|\beta} = \mu^\nu_\alpha (V^*_{\nu\|\beta} - b_{\nu\beta} V^{*3}) \quad . \tag{81}$$

Similarly,

$$V^{\alpha}|_{\beta} = (\mu^{-1})^{\alpha}_{\nu} \, (V^{*\nu}\|_{\beta} - b^{\nu}_{\beta} V^{*3}) \quad . \tag{82}$$

For the other components of $V_i|_j$ and $V^i|_j$, we obtain, in a similar manner, the following expressions (16)

$$V_{\alpha}|_3 = \mu^{\nu}_{\alpha} V^{*}_{\nu,3} \quad , \quad V^{\alpha}|_3 = (\mu^{-1})^{\alpha}_{\nu} V^{*\nu}_{\ ,3} \quad ,$$

$$V_3|_{\alpha} = V^{*}_{3,\alpha} + b^{\lambda}_{\alpha} V^{*}_{\lambda} \quad , \quad V^3|_{\alpha} = V^{*3}_{\ ,\alpha} + b_{\alpha\lambda} V^{*\lambda} \quad , \tag{83}$$

$$V^3|_3 = V_3|_3 = V_{3,3} = V^{*}_{3,3} = V^{*3}_{\ ,3} \quad .$$

Equations (81), (82), and (83) are the relations between the space and surface derivatives of a vector.

2.5 Modified Hellinger-Reissner Variational Principle

For an elastic body (18)

$$S^{ij} = \frac{1}{2} \left(\frac{\partial \Phi}{\partial \gamma_{ij}} - \frac{\partial \Phi}{\partial \gamma_{ji}} \right) \quad , \tag{84}$$

where Φ is the strain energy function, measured per unit volume of the undeformed body, and satisfying the following relation

$$\delta \Phi = S^{ij} \, \delta \gamma_{ij} \quad , \tag{85}$$

where δ stands for variation.

The modified Hellinger-Reissner variational theorem can be stated as follows: The state of stress and displacement which satisfied the differential equations of motion and the strain-

displacement relations (39) in the interior of the body, and the conditions of prescribed stress on the surface of the body, is determined by the variational equation (14)

$$\delta \left[\int_{\circ\tau} (s^{ij} \gamma_{ij} - \Phi) \, d\tau - \int_{\circ\tau} \tfrac{1}{2} s^{ij} (V_i|_j + V_j|_i + V^r|_i V_r|_j) d\tau + \int_{\circ\tau} \rho_\circ ({}_\circ b^i - {}_\circ a^i) V_i \, d\tau + \int_{\circ S} {}_\circ n_j \tilde{t}^{ji} V_i \, dS \right] = 0 \,, \tag{86}$$

in which the components of the strain tensor γ_{ij} , stress tensor s^{ij} , and displacement V_i are allowed to vary independently. The $\rho_\circ$ and ${}_\circ b^i$ denote respectively the density and the components of body force per unit mass of the undeformed body, ${}_\circ a^i$ the components of the acceleration vector, ${}_\circ\tau$ the volume of the undeformed body, ${}_\circ S$ its total boundary, where only the stress, $\tilde{t}^{ij}$ is prescribed, and $d\tau$ and dS represent the corresponding elements of volume and area, respectively.

By Equations (18), (51), and (66) the element of volume is

$$\begin{aligned} d\tau &= \sqrt{g/a} \cdot \sqrt{a} \; d\theta^1 d\theta^2 d\theta^3 \\ &= \mu \, dA \, d\theta^3 \quad . \end{aligned} \tag{87}$$

For the face boundary, the element of area is

$$dS = \mu \, dA \quad , \tag{88}$$

and, for the edge boundary (16),

$$ {}_\circ n_\alpha \, dS = \mu \, {}_\circ\nu_\alpha \, ds \, d\theta^3 \quad , \tag{89}$$

where ${}_o\nu_\alpha$ are the components of the unit vector normal to the intersection of the undeformed middle surface and the edge boundary, and ds is the line element of that intersection.

The variational Equation (86) can be written as

$$\delta\Big\{\int_{{}_o\tau}(S^{ij}\gamma_{ij}-\Phi)d\tau-\int_{{}_o\tau}\Big[\frac{S^{\alpha\beta}}{2}(V_{\alpha|\beta}+V_{\beta|\alpha}+V^{\delta}|_{\alpha}V_{\delta|\beta}+$$
$$+V^{3}|_{\alpha}V_{3|\beta})+2\frac{S^{\alpha 3}}{2}(V_{\alpha|3}+V_{3|\alpha}+V^{\delta}|_{\alpha}V_{\delta|3}+V^{3}|_{\alpha}V_{3|3})+$$
$$+\frac{S^{33}}{2}(V_{3|3}+V_{3|3}+V^{\delta}|_{3}V_{\delta|3}+V_{3|3}V_{3|3})\Big]d\tau+\int_{{}_o\tau}\Big[\rho_o({}_ob^{\alpha}+$$
$$+{}_oa^{\alpha})V_{\alpha}+\rho_o({}_ob^{3}-{}_oa^{3})V_3\Big]d\tau+\int_{{}_oS}({}_on_j\tilde{t}^{j3}V_3+$$
$$+{}_on_j\tilde{t}^{j\alpha}V_\alpha)dS\Big\}=0 \qquad . \qquad (90)$$

By using (81), (82), and (83) we can formulate Equation (90) in terms of the "shifted" components of displacement V^*_i and V^{*i} as follows.

$$\delta\Big[\int_{{}_o\tau}(S^{ij}\gamma_{ij}-\Phi)d\tau-\int_{{}_o\tau}\Big\{\frac{1}{2}S^{\alpha\beta}\big[\mu^{\delta}_{\alpha}(V^*_{\delta\|\beta}-b_{\delta\beta}V^*_3)+$$
$$+\mu^{\delta}_{\beta}(V^*_{\delta\|\alpha}-b_{\delta\alpha}V^*_3)+(V^{*\delta}\|_{\alpha}-b^{\delta}_{\alpha}V^*_3)(V^*_{\delta\|\beta}-b_{\delta\beta}V^*_3)+$$
$$+(V^*_{3,\alpha}+b^{\delta}_{\alpha}V^*_{\delta})(V^*_{3,\beta}+b^{\lambda}_{\beta}V^*_{\lambda})\big]+S^{\alpha 3}\big[\mu^{\delta}_{\alpha}V^*_{\delta,3}+$$
$$+V^*_{3,\alpha}+b^{\delta}_{\alpha}V^*_{\delta}+(V^{*\delta}\|_{\alpha}-b^{\delta}_{\alpha}V^*_3)+(V^*_{3,\alpha}+b^{\delta}_{\alpha}V^*_{\delta})V^*_{3,3}\big]+$$
$$+\big[\frac{S^{33}}{2}(2V^*_{3,3}+V^{*\delta}{}_{,3}V^*_{\delta,3}+V^*_{3,3}V^*_{3,3})\big]\Big\}d\tau+$$
$$+\int\big[\rho_o({}_ob^{\alpha}-{}_oa^{\alpha})\mu^{\delta}_{\alpha}V^*_{\delta}+\rho_o({}_ob^{3}-{}_oa^{3})V^*_3\big]d\tau+\int_{{}_oS}\big[({}_on_\beta\tilde{t}^{\beta\alpha}+$$
$$+{}_on_3\tilde{t}^{3\alpha})\mu^{\delta}_{\alpha}V^*_{\delta}+({}_on_\beta\tilde{t}^{\beta 3}+{}_on_3\tilde{t}^{33})V^*_3\big]dS\Big]=0 \qquad . \qquad (91)$$

We now carry out the indicated variation in (91), using Green's transformation and Equation (87), and also dividing the resulting volume integral into three parts, to obtain.

$$\delta I_1 + \delta I_2 + \delta I_3 + \delta J = 0 \qquad , \qquad (92)$$

where

$$\begin{aligned}\delta I_1 = \int_{\circ\tau} \Big\{ \Big\{ & [\mu \mu^\delta_\alpha s^{\alpha\beta} + \mu s^{\alpha\beta}(V^{*\delta}\|_\alpha - b^\delta_\alpha V^*_3) + \mu s^{\beta 3} V^{*\delta}{}_{,3}]\|_\beta + \\ & + [\mu s^{\alpha\beta}(V^*_{3,\alpha} + b^\lambda_\alpha V^*_\lambda) + \mu s^{\beta 3} + \mu s^{\beta 3} V^*_{3,3}]\, b^\delta_\beta \\ & - [\mu \mu^\delta_\alpha s^{\alpha 3} + \mu s^{\alpha\beta}(V^{*\delta}\|_\alpha - b^\delta_\alpha V^*_3) + \mu (s^{33} V^{*\delta}{}_{,3})]_{,3} + \\ & + \mu\, \rho_\circ \mu^\delta_\alpha ({}_\circ b^\alpha - {}_\circ a^\alpha) \Big\}\, \delta V^*_\delta + \Big\{ [\mu \mu^\delta_\alpha s^{\alpha\beta} + \\ & + \mu s^{\alpha\beta}(V^{*\delta}\|_\alpha - b^\delta_\alpha V^*_3) + \mu s^{\beta 3} V^{*\delta}{}_{,3}]\, b_{\delta\beta} + \\ & + [\mu s^{\alpha\beta}(V^*_{3,\alpha} + b^\delta_\alpha V^*_\delta) + \mu s^{\beta 3}(1 + V^*_{3,3})]\|_\beta + \\ & + [\mu s^{\alpha 3}(V^*_{3,\alpha} + b^\delta_\alpha V^*_\delta) + \mu s^{33}(1 + V^*_{3,3})]_{,3} + \\ & + \mu\, \rho_\circ ({}_\circ b^3 - {}_\circ a^3) \Big\}\, \delta V^*_3 \Big\}\, dA\, d\theta^3 \end{aligned} \qquad , \qquad (93a)$$

$$\begin{aligned}\delta I_2 = \int_{\circ\tau} \Big\{ \Big\{ & \gamma_{\alpha\beta} - \tfrac{1}{2}[\mu^\delta_\alpha (V^*_\delta\|_\beta - b_{\delta\beta} V^*_3) + \mu^\delta_\beta (V^*_\delta\|_\alpha - b_{\delta\alpha} V^*_3) + \\ & + (V^{*\delta}\|_\alpha - b^\delta_\alpha V^*_3)(V^*_\delta\|_\beta - b_{\delta\beta} V^*_3) + (V^*_{3,\alpha} + b^\delta_\alpha V^*_\delta)(V^*_{3,\beta} + \\ & + b^\delta_\beta V^*_\delta)] \Big\}\, \delta s^{\alpha\beta} + \Big\{ 2\gamma_{\alpha 3} - [\mu^\beta_\alpha V^*_{\beta,3} + V^*_{3,\alpha} + \\ & + b^\delta_\alpha V^*_\delta + (V^{*\delta}\|_\alpha + b^\delta_\alpha V^*_3)\, V^*_{\delta,3} + (V^*_{3,\alpha} + \end{aligned}$$

$$+ b^{\delta}_{\alpha} V^{*}_{\delta}) V^{*}_{3,3}] \Big\} \delta S^{\alpha 3} + \Big\{ \gamma_{33} - \tfrac{1}{2}(2V^{*}_{3,3} +$$

$$+ V^{*\delta}_{,3} V^{*}_{\delta,3} + V^{*}_{3,3} V^{*}_{3,3})\Big\} \delta S^{33} \Big\} \mu \, dA \, d\theta^{3} \quad , \qquad (93b)$$

$$\delta I_3 = \int_{{}_{\circ}\tau} \left[S^{ij} - \frac{1}{2}\left(\frac{\partial \Phi}{\partial \gamma_{ij}} - \frac{\partial \Phi}{\partial \gamma_{ji}}\right)\right] \delta \gamma_{ij} \, \mu \, dA \, d\theta^{3} \quad , \qquad (93c)$$

$$\delta J = \int_{{}_{\circ}C}\int_{-h}^{h} \Big\{ {}_{\circ}\nu_{\beta} [\mu \mu^{\delta}_{\alpha} \tilde{t}^{\beta\alpha} - \mu S^{\alpha\beta} (\mu^{\delta}_{\alpha} + V^{*\delta}\|_{\alpha} - b^{\delta}_{\alpha} V^{*}_{3})$$

$$- \mu S^{\alpha 3} V^{*\delta}_{,3}] \delta V^{*}_{\delta} + {}_{\circ}\nu_{\beta} [\mu \tilde{t}^{\beta 3} - \mu S^{\alpha\beta} (V^{*}_{3,\alpha} +$$

$$+ b^{\delta}_{\alpha} V^{*}_{\delta}) - \mu S^{\alpha 3} (1 + V^{*}_{3,3})] \delta V^{*}_{3} \Big\} d\theta^{3} ds \; +$$

$$+ \int_{{}_{\circ}A} \Big\{ {}_{\circ}n_{3} [\mu \mu^{\delta}_{\alpha} \tilde{t}^{3\alpha} - \mu S^{\alpha 3} (\mu^{\delta}_{\alpha} + V^{*\delta}\|_{\alpha} - b^{\delta}_{\alpha} V^{*}_{3}) -$$

$$- \mu S^{33} V^{*\delta}_{,3}] \delta V^{*}_{\delta} + {}_{\circ}n_{3} [\mu \tilde{t}^{33} - \qquad (93d)$$

$$- \mu S^{\alpha 3} (V^{*}_{3,\alpha} + b^{\delta}_{\alpha} V^{*}_{\delta}) - \mu S^{33} (1 +$$

$$+ V^{*}_{3,3})] \delta V^{*}_{3} \Big\} dA$$

Here ${}_{\circ}C$ represents the intersection of the undeformed reference surface the edge boundary, and ${}_{\circ}A$ stands for the total area of the undeformed reference surface.

Since the displacements, strains, and stresses are regarded as arbitrary functions and allowed to vary independently, we can set δI_1 , δI_2 , δI_3 , and δJ equal to zero, respectively, and thus obtain the four sets of fundamental equations. In what follows,

we shall apply this principle to the three-layered sandwich shell in order to obtain the corresponding equations of motion, strain-displacement relations, constitutive equations, and boundary conditions.

CHAPTER III

GENERAL NONLINEAR THEORY OF SANDWICH SHELLS

3.1 Change of Reference Surface

Let $2'h$, $2\bar{h}$, $2''h$ be the thickness of the upper facing, core, and lower facing, respectively. From now on, the single prime, bar, and double prime are employed to designate the quantities referring to the upper facing, core, and lower facing, respectively. We choose our right-handed general convected coordinates θ^i as a set of geodesic normal coordinate system situated on the middle surface of the core. It is found advantageous to choose similar coordinate systems, $'\theta^i$ and $''\theta^i$, in the middle surfaces of the upper and lower facings, respectively. Then, by the definition of geodesic normal coordinates, we have the following relations

$$\theta^3 = {}'\theta^3 + {}'z = {}''\theta^3 + {}''z \, , \quad \theta^\alpha = {}'\theta^\alpha = {}''\theta^\alpha \, . \tag{94}$$

where

$$\begin{aligned} &{}'z = \bar{h} + {}'h \, , \quad &{}''z = -(\bar{h} + {}''h) \, , \\ &-{}'h \leq {}'\theta^3 \leq {}'h \, , \quad &-{}''h \leq {}''\theta^3 \leq {}''h \, . \end{aligned} \tag{95}$$

Also let $'V_i$, $\bar{V}_i$, $''V_i$ be the components of the displacement vectors in the upper facing, core, and lower facing,

respectively. By using Equations (79) the "shifted" components of ${}'V_i$, $\bar{V}_i$, and ${}^{\bullet}V_i$ take the following form.

$$\begin{Bmatrix} {}'V_\alpha \\ \bar{V}_\alpha \\ {}''V_\alpha \end{Bmatrix} = \mu_\alpha^\beta \begin{Bmatrix} {}'V_\beta^* \\ \bar{V}_\beta^* \\ {}''V_\beta^* \end{Bmatrix} \qquad \begin{Bmatrix} {}'V_3 \\ \bar{V}_3 \\ {}''V_3 \end{Bmatrix} = \begin{Bmatrix} {}'V_3^* \\ \bar{V}_3^* \\ {}''V_3^* \end{Bmatrix} \tag{96}$$

In general ${}'V_i^*$, $\bar{V}_i^*$, and ${}''V_i^*$ are functions of θ^i , and could be expressed as an infinite power series in terms of the thickness coordinate. But, for the sake of simplicity, we shall assume that they vary linearly as follows.

$${}'V_i^* = {}'u_i + (\theta^3 - {}'z)\,{}'\Psi_i \quad , \tag{97a}$$

$$\bar{V}_i^* = \bar{u}_i + \theta^3 \bar{\Psi}_i \quad , \tag{97b}$$

$${}''V_i^* = {}''u_i + (\theta^3 - {}''z)\,{}''\Psi_i \quad . \tag{97c}$$

Substituting Equations (92) and (95) into (97), and considering the continuity conditions of the displacements at the interfaces, we obtain

$${}'u_i = \bar{u}_i + \bar{h}\bar{\Psi}_i + {}'h\,{}'\Psi_i \quad , \tag{98a}$$

$${}''u_i = \bar{u}_i - \bar{h}\bar{\Psi}_i - {}''h\,{}''\Psi_i \quad . \tag{98b}$$

Elimination of ${}'u_i$ and ${}''u_i$ from Equations (97), by using Equations (98), yields

$${}'V_i^* = {}'v_i + \theta^3\, {}'\psi_i \quad , \tag{99a}$$

$$\bar{V}_i^* = \bar{v}_i + \theta^3\, \bar{\psi}_i \quad , \tag{99b}$$

$${}''V_i^* = {}''v_i + \theta^3\, {}''\psi_i \quad , \tag{99c}$$

where

$${}'v_i = \bar{u}_i + \bar{h}(\bar{\psi}_i - {}'\psi_i) \quad , \tag{100a}$$

$$\bar{v}_i = \bar{u}_i \quad , \tag{100b}$$

$${}''v_i = \bar{u}_i - \bar{h}(\bar{\psi}_i - {}''\psi_i) \quad . \tag{100c}$$

Equations (99), imply the following strain distribution for the upper facing

$${}'\gamma_{\alpha\beta} = {}_0'\gamma_{\alpha\beta} + \theta^3\, {}_1'\gamma_{\alpha\beta} + (\theta^3)^2\, {}_2'\gamma_{\alpha\beta} \quad , \tag{101a}$$

$${}'\gamma_{\alpha 3} = {}_0'\gamma_{\alpha 3} + \theta^3\, {}_1'\gamma_{\alpha 3} \quad , \tag{101b}$$

$${}'\gamma_{33} = {}_0'\gamma_{33} \quad . \tag{101c}$$

Similar distributions for the core and lower facing corresponding to Equations (99b) and (99c) can be obtained by replacing the prime by a bar and a double prime, respectively.

3.2 Equations of Motion

We now write the volume integral δI_1 , of (93a) for each layer, consider Equations (99) - (101), and integrate with respect

to θ^3 through the thickness of the undeformed composite shell, to obtain

$$\delta I_1 = \delta' I_1 + \delta \bar{I}_1 + \delta'' I_1 \qquad , \tag{102}$$

where

$$\begin{aligned}
\delta' I_1 = \int \Big\{ \Big\{ & [\,'n^{\alpha\beta}(\delta^\delta_\beta + {'v^\delta}\|_\beta - b^\delta_\beta\, {'v_3}) + {'m^{\alpha\beta}}({'\psi^\delta}\|_\beta - b^\delta_\beta - b^\delta_\beta\, {'\psi_3}) + \\
& + {'q^\alpha}\, {'\psi^\delta}\,]\|_\alpha - [\,'n^{\alpha\beta}({'v_{3,\beta}} + b^\lambda_\beta\, {'v_\lambda}) + {'m^{\alpha\beta}}({'\psi_{3,\beta}} + \\
& + b^\lambda_\beta\, {'\psi_\lambda}) + {'q^\alpha}(1 + {'\psi_3})]\, b^\delta_\alpha + {'p^\delta} - f^\delta \Big\}\, \delta {'v_\delta} + \\
& + \Big\{ [\,'m^{\alpha\beta}(\delta^\delta_\beta + {'v^\delta}\|_\beta - b^\delta_\beta\, {'v_3}) + {'k^{\alpha\beta}}({'\psi^\delta}\|_\beta - b^\delta_\beta - \\
& - b^\delta_\beta\, {'\psi_3})]\|_\alpha - [\,'m^{\alpha\beta}({'v_{3,\beta}} + b^\lambda_\beta\, {'v_\lambda}) + {'k^{\alpha\beta}}({'\psi_{3,\beta}} + \\
& + b^\lambda_\beta\, {'\psi_\lambda})]\, b^\delta_\alpha + {'t^\alpha}\|_\alpha\, {'\psi^\delta} - {'q^\alpha}(\delta^\delta_\alpha + {'v^\delta}\|_\alpha - b^\delta_\alpha\, {'v_3}) - \\
& - {'n^{33}}\, {'\psi^\delta} + {'c^\delta} - {'m^\delta} \Big\}\, \delta {'\psi_\delta} + \Big\{ [\,'n^{\alpha\beta}({'v_{3,\beta}} + b^\delta_\beta\, {'v_\delta}) + \\
& + {'m^{\alpha\beta}}({'\psi_{3,\beta}} + b^\delta_\beta\, {'\psi_\delta}) + {'q^\alpha}(1 + {'\psi_3})]\|_\alpha + [\,'n^{\beta\delta}(\delta^\alpha_\delta + \\
& + {'v^\alpha}\|_\delta - b^\alpha_\delta\, {'v_3}) + {'m^{\beta\delta}}({'\psi^\alpha}\|_\delta - b^\alpha_\delta - b^\alpha_\delta\, {'\psi_3}) + \\
& + {'q^\beta}\, {'\psi^\alpha}]\, b_{\alpha\beta} + {'p^3} - {'f^3} \Big\}\, \delta {'v_3} + \Big\{ [\,'m^{\alpha\beta}({'v_{3,\beta}} + \\
& + b^\delta_\beta\, {'v_\delta}) + {'k^{\alpha\beta}}({'\psi_{3,\beta}} + b^\delta_\beta\, {'\psi_\delta})]\|_\alpha + [\,'m^{\beta\delta}(\delta^\alpha_\delta + \\
& + {'v^\alpha_\delta} - b^\alpha_\delta\, {'v_3}) + {'k^{\beta\delta}}({'\psi^\alpha}\|_\delta - b^\alpha_\delta - b^\alpha_\delta\, {'\psi_3})]\, b_{\alpha\beta} - \\
& - {'q^\alpha}({'v_{3,\alpha}} + b^\beta_\alpha\, {'v_\beta}) + ({'t^\alpha}\|_\alpha - {'n^{33}})(1 + {'\psi_3}) + \\
& + {'c^3} - {'m^3} \Big\}\, \delta {'\psi_3} \Big\}\, dA
\end{aligned} \tag{103}$$

Expressions for $\delta\bar{I}_1$, and $\delta\overset{*}{I}_1$, can be obtained by replacing the prime by a bar and a double prime, respectively.

The various stress, moment, body force, and acceleration resultants appearing in (103) are defined as follows.

$$'n^{\alpha\beta} = 'n^{\beta\alpha} = \int_{'z-'h}^{'z+'h} \mu\, 's^{\alpha\beta} d\theta^3 , \qquad 'q^{\alpha} = \int_{'z-'h}^{'z+'h} \mu\, 's^{\alpha 3} d\theta^3 ,$$

$$'m^{\alpha\beta} = 'm^{\beta\alpha} = \int_{'z-'h}^{'z+'h} \mu\, 's^{\alpha\beta} \theta^3 d\theta^3 , \qquad 't^{\alpha} = \int_{'z-'h}^{'z+'h} \mu\, 's^{\alpha 3} \theta^3 d\theta^3 ,$$

$$'k^{\alpha\beta} = 'k^{\beta\alpha} = \int_{'z-'h}^{'z+'h} \mu\, 's^{\alpha\beta} (\theta^3)^2 d\theta^3 , \qquad 'n^{33} = \int_{'z-'h}^{'z+'h} \mu\, 's^{33}\, d\theta^3 ,$$

$$'F^{\alpha} = \int_{'z-'h}^{'z+'h} \mu\, \mu^{\alpha}_{\beta}\, \rho_0\, {}_0b^{\beta} d\theta^3 , \qquad 'F^{3} = \int_{'z-'h}^{'z+'h} \mu\, \rho_0\, {}_0b^{3} d\theta^3 ,$$

$$'M^{\alpha} = \int_{'z-'h}^{'z+'h} \mu\, \mu^{\alpha}_{\beta}\, \rho_0\, {}_0b^{\beta} \theta^3 d\theta^3 , \qquad 'M^{3} = \int_{'z-'h}^{'z+'h} \mu\, \rho_0\, {}_0b^{3} \theta^3 d\theta^3 ,$$

$$'f^{\alpha} = \int_{'z-'h}^{'z+'h} \mu\, \mu^{\alpha}_{\beta}\, \rho_0\, {}_0a^{\beta} d\theta^3 , \qquad 'f^{3} = \int_{'z-'h}^{'z+'h} \mu\, \rho_0\, {}_0a^{3} d\theta^3 ,$$

$$'m^{\alpha} = \int_{'z-'h}^{'z+'h} \mu\, \mu^{\alpha}_{\beta}\, \rho_0\, {}_0a^{\beta} \theta^3 d\theta^3 , \qquad 'm^{3} = \int_{'z-'h}^{'z+'h} \mu\, \rho_0\, {}_0a^{3} \theta^3 d\theta^3 ,$$

$$'p^{i} = 'F^{i} + 'p^{i}_{+} - 'p^{i}_{-} , \qquad 'c^{i} = 'M^{i} + 'c^{i}_{+} - 'c^{i}_{-} , \tag{104}$$

$$'p^{\alpha}_{\pm} = \Big\{ \mu\, 's^{3\beta} [\, \mu^{\alpha}_{\beta} + 'v^{\alpha}\|_{\beta} - b^{\alpha}_{\beta}\, 'v_3 + \theta^3 ('\psi^{\alpha}\|_{\beta} - b^{\alpha}_{\beta}\, '\psi_3)] + $$

$$ + \mu\, 's^{33}\, '\psi^{\alpha} \Big\}_{\theta^3 = 'z \pm 'h}$$

$$'p^{3}_{\pm} = \Big\{ \mu\, 's^{3\beta} [\, 'v_{3,\beta} + b^{\alpha}_{\beta}\, 'v_{\alpha} + \theta^3 ('\psi_{3,\beta} + b^{\alpha}_{\beta}\, '\psi_{\alpha})] \quad + $$

$$+ \mu\, 'S^{33}(1 + '\psi_3)\Big\}_{\theta^3 = 'z \pm 'h} ,$$

$$'C_{\pm}^{\alpha} = \Big\{ \mu\, \theta^3\, 'S^{3\beta}[\mu_\beta^\alpha + '\!v^\alpha\|_\beta - b_\beta^\alpha\, 'v_3 + \theta^3('\psi^\alpha\|_\beta - b_\beta^\alpha\, '\psi_3)] +$$

$$+ \mu\, \theta^3\, 'S^{33}\, '\psi^\alpha \Big\}_{\theta^3 = 'z \pm 'h} ,$$

$$'C_{\pm}^{3} = \Big\{ \mu\, \theta^3\, 'S^{33}('\psi_3 + 1) + \mu\, \theta^3\, 'S^{3\alpha}[\, 'v_{3,\alpha} + b_\alpha^\beta\, 'v_\beta +$$

$$+ \theta^3('\psi_{3,\alpha} + b_\alpha^\beta\, '\psi_\beta)] \Big\}_{\theta^3 = 'z \pm 'h}$$

Substituting Equations (100) into (103), and setting the volume integral δI_1 equal to zero, for arbitrary and independent variations of the twelve unknowns $\delta \bar{u}_i$, $\delta \bar{\psi}_i$, $\delta '\psi_i$, and $\delta ''\psi_i$, we obtain the following twelve equations of motion

$$\delta \bar{u}_\delta : \quad \Big\{ n^{\alpha\beta}(\delta_\beta^\delta + \bar{u}^\delta\|_\beta - b_\beta^\delta \bar{u}_3) + [\bar{m}^{\alpha\beta} + ('n^{\alpha\beta} - ''n^{\alpha\beta})](\bar{\psi}^\delta\|_\beta -$$

$$- b_\beta^\delta \bar{\psi}_3) - m^{\alpha\beta} b_\beta^\delta + ('m^{\alpha\beta} - \bar{h}\, 'n^{\alpha\beta})('\psi^\delta\|_\beta - b_\beta^\delta\, '\psi_3) +$$

$$+ (''m^{\alpha\beta} + \bar{h}\, ''n^{\alpha\beta})(''\psi^\delta\|_\beta - b_\beta^\delta\, ''\psi_3) + 'q^\alpha\, '\psi^\delta + \bar{q}^\alpha \bar{\psi}^\delta +$$

$$+ ''q^\alpha\, ''\psi^\delta \Big\}\Big\|_\alpha - \Big\{ n^{\alpha\beta}(\bar{u}_{3,\beta} + b_\beta^\lambda \bar{u}_\lambda) + [\, \bar{m}^{\alpha\beta} +$$

$$+ \bar{h}('n^{\alpha\beta} - ''n^{\alpha\beta})](\bar{\psi}_{3,\beta} + b_\beta^\lambda \bar{\psi}_\lambda) + ('m^{\alpha\beta} - \bar{h}\, 'n^{\alpha\beta})('\psi_{3,\beta} +$$

$$+ b_\beta^\lambda\, '\psi_\lambda) + (''m^{\alpha\beta} + \bar{h}\, ''n^{\alpha\beta})(''\psi_{3,\beta} + b_\beta^\lambda\, ''\psi_\lambda) + q^\alpha + 'q^\alpha\, '\psi_3 +$$

$$+ \bar{q}^\alpha \bar{\psi}_3 + ''q^\alpha\, ''\psi_3 \Big\} b_\alpha^\delta + p^\delta - f^\delta = 0 \qquad (105a)$$

$$\delta \bar{u}_3 : \quad \Big\{ n^{\alpha\beta}(\bar{u}_{3,\beta} + b_\beta^\delta \bar{u}_\delta) + [\bar{m}^{\alpha\beta} + \bar{h}('n^{\alpha\beta} - ''n^{\alpha\beta})](\bar{\psi}_{3,\beta} +$$

$$+ b^\delta_\beta \bar\psi_\delta) + ({}'m^{\alpha\beta} - \bar h\, {}'n^{\alpha\beta})({}'\psi_{3,\beta} + b^\delta_\beta\, {}'\psi_\delta) + ({}''m^{\alpha\beta} +$$

$$+ \bar h\, {}''n^{\alpha\beta})({}''\psi_{3,\beta} + b^\delta_\beta\, {}''\psi_\delta) + q^\alpha + {}'q^\alpha\, {}'\psi_3 + \bar q^\alpha \bar\psi_3 +$$

$$+ {}''q^\alpha\, {}''\psi_3 \Big\}\Big\|_\alpha + \Big\{ n^{\beta\delta}(\delta^\alpha_\delta + \bar u^\alpha\|_\delta - b^\alpha_\delta \bar u_3) +$$

$$+ [\bar m^{\beta\delta} + \bar h({}'n^{\beta\delta} - {}''n^{\beta\delta})](\bar\psi^\alpha\|_\delta - b^\alpha_\delta\bar\psi_3) - m^{\beta\delta} b^\alpha_\delta +$$

$$+ ({}'m^{\beta\delta} - \bar h\,{}'n^{\beta\delta})({}'\psi^\alpha\|_\delta - b^\alpha_\delta\,{}'\psi_3) + ({}''m^{\beta\delta} +$$

$$+ \bar h\,{}''n^{\beta\delta})({}''\psi^\alpha\|_\delta - b^\alpha_\delta\,{}''\psi_3) + {}'q^\beta\,{}'\psi^\alpha + \bar q^\beta\bar\psi^\alpha +$$

$$+ {}''q^\beta\,{}''\psi^\alpha \Big\} b_{\alpha\beta} + p^3 - f^3 = 0 \quad , \qquad (105b)$$

$\delta\bar\psi$:

$$\Big\{[\bar m^{\alpha\beta} + \bar h({}'n^{\alpha\beta} - {}''n^{\alpha\beta})](\delta^\delta_\beta + \bar u^\delta\|_\beta - b^\delta_\beta \bar u_3) - [\bar k^{\alpha\beta} + \bar h({}'m^{\alpha\beta} -$$

$$- {}''m^{\alpha\beta})] b^\delta_\beta + [\bar k^{\alpha\beta} + \bar h^2({}'n^{\alpha\beta} + {}''n^{\alpha\beta})](\bar\psi^\delta\|_\beta - b^\delta_\beta\bar\psi_3) +$$

$$+ \bar h({}'m^{\alpha\beta} - \bar h\,{}'n^{\alpha\beta})({}'\psi^\delta\|_\beta - b^\delta_\beta\,{}'\psi_3) - \bar h({}''m^{\alpha\beta} +$$

$$+ \bar h\,{}''n^{\alpha\beta})({}''\psi^\delta\|_\beta - b^\delta_\beta\,{}''\psi_3) + \bar h({}'q^\alpha\,{}'\psi^\delta - {}''q^\alpha\,{}''\psi^\delta)\Big\}\Big\|_\alpha -$$

$$- \Big\{[\bar m^{\alpha\beta} + \bar h({}'n^{\alpha\beta} - {}''n^{\alpha\beta})](\bar u_{3,\beta} + b^\lambda_\beta \bar u_\lambda) + [\bar k^{\alpha\beta} + \bar h^2({}'n^{\alpha\beta} +$$

$$+ {}''n^{\alpha\beta})](\bar\psi_{3,\beta} + b^\lambda_\beta\bar\psi_\lambda) + \bar h({}'m^{\alpha\beta} - \bar h\,{}'n^{\alpha\beta})({}'\psi_{3,\beta} +$$

$$+ b^\lambda_\beta\,{}'\psi_\lambda) - \bar h({}''m^{\alpha\beta} + \bar h\,{}''n^{\alpha\beta})({}''\psi_{3,\beta} + b^\lambda_\beta\,{}''\psi_\lambda) +$$

$$+ [{}'q^\alpha(1 + {}'\psi_3) - {}''q^\alpha(1 + {}''\psi_3)]\bar h \Big\} b^\delta_\alpha + (\bar t^\alpha\|_\alpha -$$

$$- \bar n^{33})\bar\psi^\delta - \bar q^\alpha(\delta^\delta_\alpha + \bar u^\delta\|_\alpha - b^\delta_\alpha \bar u_3) +$$

$$+ c^\delta - m^\delta = 0 \quad , \qquad (105c)$$

$\delta\bar\psi_3$:

$$\Big\{[\bar m^{\alpha\beta} + \bar h({}'n^{\alpha\beta} - {}''n^{\alpha\beta})](\bar u_{3,\beta} + b^\delta_\beta\bar u_\delta) + [\bar k^{\alpha\beta} + \bar h^2({}'n^{\alpha\beta} +$$

$$+ {}''n^{\alpha\beta})](\bar\psi_{3,\beta} + b^\delta_\beta\bar\psi_\delta) + \bar h({}'m^{\alpha\beta} - \bar h\,{}'n^{\alpha\beta})({}'\psi_{3,\beta} + b^\delta_\beta\,{}'\psi_\delta) - \bar h({}''m^{\alpha\beta} +$$

$$+ \bar h \, {}''n^{\alpha\beta}) ({}''\psi_{3,\beta} + b^{\delta}_{\beta} {}''\psi_\delta) + [{}'q^\alpha(1+{}''\psi_3) - {}''q^\alpha(1+{}'\psi_3)] \bar h \}\|_\alpha +$$

$$+ \{[\bar m^{\beta\delta} + \bar h({}'n^{\beta\delta} - {}''n^{\beta\delta})](\delta^\alpha_\delta + \bar u^\alpha\|_\delta - b^\alpha_\delta \bar u_3) + [\bar k^{\beta\delta} +$$

$$+ \bar h^2({}'n^{\beta\delta} + {}''n^{\beta\delta})](\bar\psi^\alpha\|_\delta - b^\alpha_\delta \bar\psi_3) - [\bar k^{\beta\delta} + \bar h({}'m^{\beta\delta} -$$

$$- {}''m^{\beta\delta})] b^\alpha_\delta + \bar h({}'m^{\beta\delta} - \bar h \, {}'n^{\beta\delta})({}'\psi^\alpha\|_\delta - b^\alpha_\delta {}'\psi_3) -$$

$$- \bar h({}''m^{\beta\delta} + \bar h \, {}''n^{\beta\delta})({}''\psi^\alpha\|_\delta - b^\alpha_\delta {}''\psi_3) + ({}'q^\beta {}''\psi^\alpha -$$

$$- {}''q^\beta {}'\psi^\alpha)\bar h \} b_{\alpha\beta} - \bar q^\alpha(\bar u_{3,\alpha} + b^\beta_\alpha \bar u_\beta) +$$

$$+ (\bar t^\alpha\|_\alpha - \bar n^{33})(1+\bar\psi_3) + C^3 - m^3 = 0 \quad , \qquad (105d)$$

$\delta {}'\psi_\delta$:

$$\{({}'m^{\alpha\beta} - \bar h \, {}'n^{\alpha\beta})[\delta^\delta_\beta + \bar u^\delta\|_\beta + \bar h(\bar\psi^\delta\|_\beta - {}'\psi^\delta\|_\beta) -$$

$$- b^\delta_\beta \bar u_3 - b^\delta_\beta \bar h(\bar\psi_3 - {}'\psi_3)] + ({}'k^{\alpha\beta} - \bar h \, {}'m^{\alpha\beta})[{}'\psi^\delta\|_\beta -$$

$$- b^\delta_\beta(1+{}'\psi_3)] - \bar h \, {}'q^\alpha {}'\psi^\delta \}\|_\alpha - \{({}'m^{\alpha\beta} - \bar h \, {}'n^{\alpha\beta})[\bar u_{3,\beta} +$$

$$+ \bar h(\bar\psi_{3,\beta} - {}'\psi_{3,\beta}) + b^\lambda_\beta \bar u_\lambda + \bar h \, b^\lambda_\beta(\bar\psi_\lambda - {}'\psi_\lambda)] -$$

$$- ({}'k^{\alpha\beta} - \bar h \, {}'m^{\alpha\beta})({}'\psi_{3,\beta} + b^\lambda_\beta {}'\psi_\lambda) - {}'q^\alpha \bar h(1+$$

$$+ {}'\psi_3)\} b^\delta_\alpha + ({}'t^\alpha\|_\alpha - {}'n^{33}) {}'\psi^\delta - {}'q^\alpha[\delta^\delta_\alpha +$$

$$+ \bar u^\delta\|_\alpha + \bar h(\bar\psi^\delta\|_\alpha - {}'\psi^\delta\|_\alpha) - b^\delta_\alpha \bar u_3 - \bar h \, b^\delta_\alpha(\bar\psi_3 -$$

$$- {}'\psi_3)] + {}'g^\delta - {}'d^\delta = 0 \quad , \qquad (105e)$$

$\delta {}'\psi_3$:

$$\{({}'m^{\alpha\beta} - \bar h \, {}'n^{\alpha\beta})[\bar u_{3,\beta} + \bar h(\bar\psi_{3,\beta} - {}'\psi_{3,\beta}) + b^\delta_\beta \bar u_\delta + \bar h \, b^\delta_\beta(\bar\psi_\delta -$$

$$- {}'\psi_\delta)] + ({}'k^{\alpha\beta} - \bar h \, {}'m^{\alpha\beta})({}'\psi_{3,\beta} + b^\delta_\beta {}'\psi_\delta) - {}'q^\alpha \bar h(1+$$

$$+ {}'\psi_3)\}\|_\alpha + \{({}'m^{\beta\delta} - \bar h \, {}'n^{\beta\delta})[\delta^\alpha_\delta + \bar u^\alpha\|_\delta + \bar h(\bar\psi^\alpha\|_\delta -$$

$$- {}'\psi^\alpha\|_\delta) - b^\alpha_\delta \bar u_3 - \bar h \, b^\alpha_\delta(\bar\psi_3 - {}'\psi_3) + ({}'k^{\beta\delta} -$$

$$- \bar{h}\,'m^{\beta\delta})\,[\,'\psi^{\alpha}\|_{\delta} - b^{\alpha}_{\delta}(1 + '\psi_3)] - \bar{h}\,'q^{\beta}\,'\psi^{\alpha}\Big\}\, b_{\alpha\beta} -$$

$$- 'q^{\alpha}[\,\bar{u}_{3,\alpha} + \bar{h}(\bar{\psi}_{3,\alpha} - '\psi_{3,\alpha}) + b^{\beta}_{\alpha}\bar{u}_{\beta} + \bar{h}\, b^{\beta}_{\alpha}(\bar{\psi}_{\beta} -$$

$$- '\psi_{\beta})] + ('t^{\alpha}\|_{\alpha} - 'n^{33})(1 + '\psi_3) + 'g^{3} - 'd^{3} = 0\ , \qquad \text{(105f)}$$

$$\delta\, ''\psi_{\delta}: \quad \Big\{(''m^{\alpha\beta} + \bar{h}\, ''n^{\alpha\beta})\,[\,\delta^{\delta}_{\beta} + \bar{u}^{\delta}\|_{\beta} - \bar{h}(\bar{\psi}^{\delta}\|_{\beta} - ''\psi^{\delta}\|_{\beta}) -$$

$$- b^{\delta}_{\beta}\bar{u}_3 + \bar{h}\, b^{\delta}_{\beta}(\bar{\psi}_3 - ''\psi_3)] + (''k^{\alpha\beta} + \bar{h}\, ''m^{\alpha\beta})\,[\,''\psi^{\delta}\|_{\beta} -$$

$$- b^{\delta}_{\beta}(1 + ''\psi_3)] + \bar{h}\, ''q^{\alpha}\, ''\psi^{\delta}\Big\}\Big\|_{\alpha} - \Big\{(''m^{\alpha\beta} +$$

$$+ \bar{h}\, ''n^{\alpha\beta})\,[\,\bar{u}_{3,\alpha} - \bar{h}(\bar{\psi}_{3,\alpha} - ''\psi_{3,\alpha}) + b^{\lambda}_{\alpha}\bar{u}_{\lambda} -$$

$$- \bar{h}\, b^{\lambda}_{\alpha}(\bar{\psi}_{\lambda} - ''\psi_{\lambda})] + (''k^{\alpha\beta} + \bar{h}\, ''m^{\alpha\beta})\,(''\psi_{3,\alpha} +$$

$$+ b^{\lambda}_{\alpha}\, ''\psi_{\lambda}) + \bar{h}\, ''q^{\beta}(1 + ''\psi_3)\Big\}\, b^{\delta}_{\beta} + (''t^{\alpha}\|_{\alpha} -$$

$$- ''n^{33})\, ''\psi^{\delta} - ''q^{\alpha}[\,\delta^{\delta}_{\alpha} + \bar{u}^{\delta}\|_{\alpha} + \bar{h}(\bar{\psi}^{\delta}\|_{\alpha} - ''\psi^{\delta}\|_{\alpha}) -$$

$$- b^{\delta}_{\alpha}\bar{u}_3 - \bar{h}\, b^{\delta}_{\alpha}(\bar{\psi}_{\delta} - ''\psi_{\delta})] + ''g^{\delta} + ''d^{\delta} = 0\ , \qquad \text{(105g)}$$

$$\delta\, ''\psi_3: \quad \Big\{(''m^{\alpha\beta} + \bar{h}\, ''n^{\alpha\beta})\,[\,\bar{u}_{3,\beta} - \bar{h}(\bar{\psi}_{3,\beta} - ''\psi_{3,\beta}) + b^{\delta}_{\beta}\bar{u}_{\delta} -$$

$$- \bar{h}\, b^{\delta}_{\beta}(\bar{\psi}_{\delta} - ''\psi_{\delta})] + (''k^{\alpha\beta} + \bar{h}\, ''m^{\alpha\beta})\,(''\psi_{3,\beta} +$$

$$+ b^{\delta}_{\beta}\, ''\psi_{\delta}) + ''q^{\alpha}\bar{h}(1 + ''\psi_3)\Big\}\Big\|_{\alpha} + \Big\{(''m^{\beta\delta} +$$

$$+ \bar{h}\, ''n^{\beta\delta})\,[\,\delta^{\alpha}_{\delta} + \bar{u}^{\alpha}\|_{\delta} - \bar{h}(\bar{\psi}^{\alpha}\|_{\delta} - ''\psi^{\alpha}\|_{\delta}) -$$

$$- b^{\alpha}_{\delta}\bar{u}_3 + \bar{h}\, b^{\alpha}_{\delta}(\bar{\psi}_3 - ''\psi_3) + (''k^{\beta\delta} + \bar{h}\, ''m^{\beta\delta})\,[\,''\psi^{\alpha}\|_{\delta} -$$

$$- b^{\alpha}_{\delta}(1 + ''\psi_3)] + \bar{h}\, ''q^{\beta}\, ''\psi^{\alpha}\Big\}\, b_{\alpha\beta} - ''q^{\alpha}[\,\bar{u}_{3,\alpha} - \bar{h}(\bar{\psi}_{3,\alpha} -$$

$$- ''\psi_{3,\alpha}) + b^{\beta}_{\alpha}\bar{u}_{\beta} - \bar{h}\, b^{\beta}_{\alpha}(\bar{\psi}_{\beta} - ''\psi_{\beta})] +$$

$$+ ({}^{\prime\prime}t^{\alpha}\|_{\alpha} - {}^{\prime}n^{33})(1 + {}^{\prime\prime}\psi_3) + {}^{\prime}g^3 + {}^{\prime}d^3 = 0 \quad , \tag{105h}$$

where

$$n^{\alpha\beta} = {}^{\prime}n^{\alpha\beta} + \bar{n}^{\alpha\beta} + {}^{\prime\prime}n^{\alpha\beta} \quad ,$$

$$m^{\alpha\beta} = {}^{\prime}m^{\alpha\beta} + \bar{m}^{\alpha\beta} + {}^{\prime\prime}m^{\alpha\beta} \quad ,$$

$$q^{\delta} = {}^{\prime}q^{\delta} + \bar{q}^{\delta} + {}^{\prime\prime}q^{\delta} \quad ,$$

$$F^{i} = {}^{\prime}F^{i} + \bar{F}^{i} + {}^{\prime\prime}F^{i} \quad ,$$

$$p^{i} = {}^{\prime}p^{i} + \bar{p}^{i} + {}^{\prime\prime}p^{i} \quad ,$$

$$= F^{i} + {}^{\prime}p^{i}_{+} - {}^{\prime\prime}p^{i}_{-} \quad ,$$

$$C^{i} = \bar{M}^{i} + \bar{h}({}^{\prime}F^{i} - {}^{\prime\prime}F^{i}) + \bar{h}({}^{\prime}p^{i}_{+} + {}^{\prime\prime}p^{i}_{-}) \quad , \tag{106}$$

$${}^{\prime}g^{i} = {}^{\prime}M^{i} - \bar{h}\,{}^{\prime}F^{i} + 2\,{}^{\prime}h\,{}^{\prime}p^{i}_{+} \quad ,$$

$${}^{\prime\prime}g^{i} = {}^{\prime\prime}M^{i} - \bar{h}\,{}^{\prime\prime}F^{i} + 2\,{}^{\prime\prime}h\,{}^{\prime\prime}p^{i}_{-} \quad ,$$

$$f^{i} = {}^{\prime}f^{i} + \bar{f}^{i} + {}^{\prime\prime}f^{i} \quad ,$$

$$m^{i} = \bar{m}^{i} + \bar{h}({}^{\prime}f^{i} - {}^{\prime\prime}f^{i}) \quad ,$$

$${}^{\prime}d^{i} = {}^{\prime}m^{i} - \bar{h}\,{}^{\prime}f^{i} \quad ,$$

$${}^{\prime\prime}d^{i} = -({}^{\prime\prime}m^{i} + \bar{h}\,{}^{\prime\prime}f^{i}) \quad .$$

3.3 Strain-displacement Relations

Writing (93b) for each layer, considering Equations (99) - (101), and integrating across the thickness of the composite shell, we obtain

$$\delta I_2 = \delta' I_2 + \delta \bar{I}_2 + \delta'' I_2 \qquad , \tag{107}$$

where

$$\delta' I_2 = \int_{oA} \Big\{ \Big\{ {}_{o}\gamma'_{\alpha\beta} - \tfrac{1}{2}[('v_{\alpha\|\beta} - b_{\alpha\beta}\,'v_3) + ('v_{\beta\|\alpha} - b_{\beta\alpha}\,'v_3) +$$

$$+ ('v^{\delta}\|_{\alpha} - b^{\delta}_{\alpha}\,'v_3) ('v_{\delta\|\beta} - b_{\delta\beta}\,'v_3) + ('v_{3,\alpha} + b^{\delta}_{\alpha}\,'v_\delta) ('v_{3,\beta} +$$

$$+ b^{\lambda}_{\beta}\,'v_\lambda)] \Big\} \delta' n^{\alpha\beta} + \Big\{ {}_{1}\gamma'_{\alpha\beta} - \tfrac{1}{2}[('\psi_{\alpha\|\beta} - b_{\alpha\beta}\,'\psi_3) +$$

$$+ ('\psi_{\beta\|\alpha} - b_{\beta\alpha}\,'\psi_3) - b^{\delta}_{\alpha} ('v_{\delta\|\beta} - b_{\delta\beta}\,'v_3) - b^{\delta}_{\beta} ('v_{\delta\|\alpha} -$$

$$- b_{\beta\alpha}\,'\psi_3) - b^{\delta}_{\alpha} ('v_{\delta\|\beta} - b_{\delta\beta}\,'v_3) - b^{\delta}_{\beta} ('v_{\delta\|\alpha} - b_{\delta\alpha}\,'v_3) +$$

$$+ ('v^{\delta}\|_{\alpha} - b^{\delta}_{\alpha}\,'v_3) ('\psi_{\delta\|\beta} - b_{\delta\beta}\,'\psi_3) + ('v^{\delta}\|_{\beta} -$$

$$- b^{\delta}_{\beta}\,'v_3) + ('v^{\delta}\|_{\alpha} - b^{\delta}_{\alpha}\,'v_3) ('\psi_{\delta\|\beta} - b_{\delta\beta}\,'\psi_3) +$$

$$+ ('v^{\delta}\|_{\beta} - b^{\delta}_{\beta}\,'v_3) ('\psi_{3,\beta} + b^{\lambda}_{\beta}\,'\psi_\lambda) + ('\psi_{3,\alpha} +$$

$$+ b^{\delta}_{\alpha}\,'\psi_\delta) ('v_{3,\beta} + b^{\lambda}_{\beta}\,'v_\lambda)] \Big\} \delta' m^{\alpha\beta} + \Big\{ {}_{2}\gamma'_{\alpha\beta} -$$

$$- \tfrac{1}{2}[- b^{\delta}_{\alpha} ('\psi_{\delta\|\beta} - b_{\delta\beta}\,'\psi_3) - b^{\delta}_{\beta} ('\psi_{\delta\|\alpha} - b_{\delta\alpha}\,'\psi_3) +$$

$$+ ('\psi^{\delta}\|_{\beta} - b^{\delta}_{\beta}\,'\psi_3) ('\psi_{\delta\|\alpha} - b_{\delta\alpha}\,'\psi_3) + ('\psi_{3,\alpha} + b^{\delta}_{\alpha}\,'\psi_\delta) ('\psi_{3,\beta} +$$

$$+ b^{\nu}_{\beta}\,'\psi_\nu)] \Big\} \delta' k^{\alpha\beta} + \Big\{ 2\,{}_{o}\gamma'_{\alpha 3} - [('v_{3,\alpha} +$$

$$+ b^{\delta}_{\alpha} \overset{\prime}{v}_{\delta}) (1 + \overset{\prime}{\psi}_3) + \overset{\prime}{\psi}_{\alpha} + (\overset{\prime}{v}^{\delta} \|_{\alpha} -$$

$$- b^{\delta}_{\alpha} \overset{\prime}{v}_3) \overset{\prime}{\psi}_{\delta}] \Big\} \delta \overset{\prime}{q}^{\alpha} + \Big\{ 2 \, {}_1\overset{\prime}{\gamma}_{\alpha 3} - [\overset{\prime}{\psi}_{3,\alpha} (1 +$$

$$+ \overset{\prime}{\psi}_3) + \overset{\prime}{\psi}_{\beta} \overset{\prime}{\psi}^{\beta} \|_{\alpha}] \Big\} \delta \overset{\prime}{t}^{\alpha} + \Big\{ {}_0\overset{\prime}{\gamma}_{33} - \tfrac{1}{2} [2 \overset{\prime}{\psi}_3 +$$

$$+ \overset{\prime}{\psi}_{\alpha} \overset{\prime}{\psi}^{\alpha} + (\overset{\prime}{\psi}_3)^2] \Big\} \delta \overset{\prime}{n}^{33} \Big] dA \quad . \tag{108}$$

Expressions for $\delta \bar{I}_2$ and $\delta \overset{\prime\prime}{I}_2$ are similar to Equation (108).

Setting $\delta \overset{\prime}{I}_2$ equal to zero, the vanishing of the coefficients of the arbitrary and independent variations of the stress and couple resultants in (108) leads to the strain-displacement relations. By using (100), we obtain

$$\begin{aligned}
{}_0\overset{\prime}{\gamma}_{\alpha\beta} = \frac{1}{2} \Big\{ & [\bar{u}_{\alpha\|\beta} + \bar{h} (\bar{\psi}_{\alpha\|\beta} - \overset{\prime}{\psi}_{\alpha\|\beta}) - b_{\alpha\beta} \bar{u}_3 - \bar{h} b_{\alpha\beta} (\bar{\psi}_3 - \\
& - \overset{\prime}{\psi}_3)] + [\bar{u}_{\beta\|\alpha} + \bar{h} (\bar{\psi}_{\beta\|\alpha} - \overset{\prime}{\psi}_{\alpha\|\beta}) - b_{\beta\alpha} \bar{u}_3 - \\
& - \bar{h} b_{\alpha\beta} (\bar{\psi}_3 - \overset{\prime}{\psi}_3)] + [\bar{u}^{\delta} \|_{\alpha} + \bar{h} (\bar{\psi}^{\delta} \|_{\alpha} - \overset{\prime}{\psi}^{\delta} \|_{\alpha}) - \\
& - b^{\delta}_{\beta} \bar{u}_3 - \bar{h} b^{\delta}_{\alpha} (\bar{\psi}_3 - \overset{\prime}{\psi}_3)] [\bar{u}_{\delta\|\beta} + \bar{h} (\bar{\psi}_{\delta\|\beta} - \\
& - \overset{\prime}{\psi}_{\delta\|\beta}) - b_{\delta\alpha} \bar{u}_3 - \bar{h} b_{\delta\alpha} (\bar{\psi}_3 - \overset{\prime}{\psi}_3)] + [\bar{u}_{3,\alpha} + \\
& + \bar{h} (\bar{\psi}_{3,\alpha} - \overset{\prime}{\psi}_{3,\alpha}) + b^{\delta}_{\alpha} \bar{u}_{\delta} + \bar{h} b^{\delta}_{\alpha} (\bar{\psi}_{\delta} - \overset{\prime}{\psi}_{\delta})] [\bar{u}_{3,\beta} + \\
& + \bar{h} (\bar{\psi}_{3,\beta} - \overset{\prime}{\psi}_{3,\beta}) + b^{\lambda}_{\beta} \bar{u}_{\lambda} + \\
& + \bar{h} b^{\lambda}_{\beta} (\bar{\psi}_{\lambda} - \overset{\prime}{\psi}_{\lambda})] \Big\} \quad ,
\end{aligned} \tag{109a}$$

$$
\begin{aligned}
{}_1\gamma'_{\alpha\beta} = \frac{1}{2}\Big\{ & (\psi'_{\alpha\|\beta} - b_{\alpha\beta}\psi'_3) + (\psi'_{\beta\|\alpha} - b_{\beta\alpha}\psi'_3) - b^\delta_\alpha [\, \bar{u}_{\delta\|\beta} + \\
& + \bar{h}(\bar{\psi}_{\delta\|\beta} - \psi'_{\delta\|\beta}) - b_{\delta\beta}\bar{u}_3 - \bar{h}\, b_{\delta\beta}(\bar{\psi}_3 - \psi'_3)] - \\
& - b^\delta_\beta [\, \bar{u}_{\delta\|\alpha} + \bar{h}(\bar{\psi}_{\delta\|\alpha} - \psi'_{\delta\|\alpha}) - b_{\delta\alpha}\bar{u}_3 - \\
& - \bar{h}\, b_{\delta\alpha}(\bar{\psi}_3 - \psi'_3)] + [\, \bar{u}^\delta\|_\alpha + \bar{h}(\bar{\psi}^\delta\|_\alpha - \psi'^\delta\|_\alpha) - \\
& - b^\delta_\alpha \bar{u}_3 - \bar{h}\, b^\delta_\alpha(\bar{\psi}_3 - \psi'_3)]\,(\psi'_{\delta\|\beta} - b_{\delta\beta}\psi'_3) + \\
& + [\, \bar{u}^\delta\|_\beta + \bar{h}(\bar{\psi}^\delta\|_\alpha - \psi'^\delta\|_\alpha) - b^\delta_\beta \bar{u}_3 - \bar{h}\, b^\delta_\beta(\bar{\psi}_3 - \\
& - \psi'_3) + [\, \bar{u}_{3,\alpha} + \bar{h}(\bar{\psi}_{3,\alpha} - \psi'_{3,\alpha}) + b^\delta_\alpha \bar{u}_\delta + \\
& + \bar{h}\, b^\delta_\alpha(\bar{\psi}_\delta - \psi'_\delta)]\,(\psi'_{3,\beta} + b^\lambda_\beta \psi'_\lambda) + [\, \bar{u}_{3,\beta} + \bar{h}(\bar{\psi}_{3,\alpha} - \\
& - \psi'_{3,\alpha}) + b^\lambda_\beta \bar{u}_\lambda + \bar{h}\, b^\lambda_\beta(\bar{\psi}_\lambda - \psi'_\lambda)]\,(\psi'_{3,\alpha} + b^\delta_\alpha \psi'_\delta) \Big\} \quad ,
\end{aligned}
\qquad (109b)
$$

$$
\begin{aligned}
{}_2\gamma'_{\alpha\beta} = \frac{1}{2}\Big\{ & - b^\delta_\alpha(\psi'_{\delta\|\beta} - b_{\delta\beta}\psi'_3) - b^\delta_\beta(\psi'_{\delta\|\alpha} - b_{\delta\alpha}\psi'_3) + \\
& + (\psi'^\delta\|_\beta - b^\delta_\beta \psi'_3)(\psi'_{\delta\|\alpha} - b_{\delta\alpha}\psi'_3) + (\psi'_{3,\alpha} + \\
& + b^\delta_\alpha \psi'_\delta)(\psi'_{3,\beta} + b^\lambda_\beta \psi'_\lambda) \Big\} \quad ,
\end{aligned}
\qquad (109c)
$$

$$
\begin{aligned}
{}_0\gamma'_{\alpha 3} = \frac{1}{2}\Big\{ & [\, \bar{u}_{3,\alpha} + \bar{h}(\bar{\psi}_{3,\alpha} - \psi'_{3,\alpha}) + b^\delta_\alpha \bar{u}_\delta + \bar{h}\, b^\delta_\alpha(\bar{\psi}_\delta - \\
& - \psi'_\delta)]\,(1 + \psi'_3) + \psi'_\alpha + [\, \bar{u}^\delta\|_\alpha + \bar{h}(\bar{\psi}^\delta\|_\alpha - \\
& - \psi'^\delta\|_\alpha) - b^\delta_\alpha \bar{u}_3 - \bar{h}\, b^\delta_\alpha(\bar{\psi}_3 - \psi'_3)\,\psi'_\delta \Big\} \quad ,
\end{aligned}
\qquad (109d)
$$

$$
{}_1\gamma'_{\alpha 3} = \frac{1}{2}\,[\, \psi'_{3,\alpha}(1 + \psi'_3) + \psi'_\beta \psi'^\beta\|_\alpha] \quad , \qquad (109e)
$$

$$
{}_0\gamma'_{33} = \frac{1}{2}\,[\, 2\psi'_3 + \psi'_\alpha \psi'^\alpha + (\psi'_3)^2] \quad , \qquad (109f)
$$

The strain-displacement relations for the core and lower facing can be obtained by replacing the prime by a bar and a double prime, and letting $\bar{h} = 0$ and $\bar{h} = -\bar{h}$, respectively.

3.4 Constitutive Equations

By analogy with Equations (45), the stress-strain relations for the upper facing can be written as

$$'S^{\alpha\beta} = {}'C^{\alpha\beta\delta\nu}\,({}'\gamma_{\delta\nu} - {}'\alpha_{\delta\nu}\,{}'T) + {}'C^{\alpha\beta 33}\,({}'\gamma_{33} - {}'\alpha_{33}\,{}'T) \quad , \tag{110a}$$

$$'S^{\alpha 3} = 2\,{}'C^{\alpha 3\beta 3}\,({}'\gamma_{\beta 3} - \alpha_{\beta 3}\,{}'T) \quad , \tag{110b}$$

$$'S^{33} = {}'C^{33\delta\nu}\,({}'\gamma_{\delta\nu} - {}'\alpha_{\delta\nu}\,{}'T) + {}'C^{3333}\,({}'\gamma_{33} - {}'\alpha_{33}\,{}'T) \quad , \tag{110c}$$

Similar relations hold for the core and lower facing, respectively.

Writing the volume integral δI_3 of (93c) for each layer, considering (99) - (101), and integrating across the thickness, we obtain

$$\delta I_3 = \delta\,{}'I_3 + \delta\,\bar{I}_3 + \delta\,{}''I_3 \quad , \tag{111}$$

where

$$\delta\,{}'I_3 = \int_{oA} \Big\{ \Big[{}'n^{\alpha\beta} - \tfrac{1}{2} \Big(\frac{\partial\,{}'\Sigma}{\partial\,{}'\gamma_{o\alpha\beta}} + \frac{\partial\,{}'\Sigma}{\partial\,{}'\gamma_{o\beta\alpha}} \Big) \Big] \delta\,{}'\gamma_{o\alpha\beta} + \Big[{}'m^{\alpha\beta} -$$

$$- \tfrac{1}{2} \Big(\frac{\partial\,{}'\Sigma}{\partial\,{}'\gamma_{1\alpha\beta}} + \frac{\partial\,{}'\Sigma}{\partial\,{}'\gamma_{1\beta\alpha}} \Big) \Big] \delta\,{}'\gamma_{1\alpha\beta} + \Big[{}'k^{\alpha\beta} - \tfrac{1}{2} \Big(\frac{\partial\,{}'\Sigma}{\partial\,{}'\gamma_{2\alpha\beta}} +$$

$$+ \frac{\partial\,{}'\Sigma}{\partial\,{}'\gamma_{2\beta\alpha}} \Big) \Big] \delta\,{}'\gamma_{2\alpha\beta} + \Big[2\,{}'q^{\alpha} - \Big(\frac{\partial\,{}'\Sigma}{\partial\,{}'\gamma_{o\alpha 3}} + \frac{\partial\,{}'\Sigma}{\partial\,{}'\gamma_{o3\alpha}} \Big) \Big] \delta\,{}'\gamma_{o\alpha 3} +$$

$$+ \Big[2\,{}'t^{\alpha} - \Big(\frac{\partial\,{}'\Sigma}{\partial\,{}'\gamma_{1\alpha 3}} + \frac{\partial\,{}'\Sigma}{\partial\,{}'\gamma_{13\alpha}} \Big) \Big] \delta\,{}'\gamma_{1\alpha 3} + \Big[{}'n^{33} -$$

$$- \frac{1}{2}\left(\frac{\partial \acute{\Sigma}}{\partial\, {}_{0}\acute{\gamma}_{33}} - \frac{\partial \acute{\Sigma}}{\partial\, {}_{0}\acute{\gamma}_{33}}\right) \Big] \delta\, {}_{0}\acute{\gamma}_{33} \Big\} \, d A \quad , \tag{112a}$$

and

$$\acute{\Sigma} = \int_{\acute{z}-\acute{h}}^{\acute{z}+\acute{h}} \mu \, \acute{\Phi} \, d\theta^{3} \tag{112b}$$

is the strain energy function per unit area of the undeformed reference surface ${}_{0}A$.

The strain energy function, $\acute{\Phi}$, for the elastic anisotropic medium subjected to a prescribed steady temperature field, $\acute{T}$ is now assumed in the following form (14).

$$\acute{\Phi} = \frac{1}{2}\, \acute{S}^{ij} \left(\acute{\gamma}_{ij} - \acute{\alpha}_{ij}\, \acute{T} \right) \quad . \tag{113}$$

Substituting Equations (110) and (101) into (113) and then (112), and setting the Equation (111) equal to zero, the following constitutive equations are obtained

$$\acute{n}^{\alpha\beta} = {}_{0}\acute{B}^{\alpha\beta\delta\lambda}\, {}_{0}\acute{\gamma}_{\delta\lambda} + {}_{1}\acute{B}^{\alpha\beta\delta\lambda}\, {}_{1}\acute{\gamma}_{\delta\lambda} + {}_{2}\acute{B}^{\alpha\beta\delta\lambda}\, {}_{1}\acute{\gamma}_{\delta\lambda} +$$
$$+ {}_{0}\acute{B}^{\alpha\beta 33}\, {}_{0}\acute{\gamma}_{33} - {}_{0}\acute{T}^{\alpha\beta} \quad , \tag{114a}$$

$$\acute{m}^{\alpha\beta} = {}_{1}\acute{B}^{\alpha\beta\delta\lambda}\, {}_{0}\acute{\gamma}_{\delta\lambda} + {}_{2}\acute{B}^{\alpha\beta\delta\lambda}\, {}_{1}\acute{\gamma}_{\delta\lambda} + {}_{3}\acute{B}^{\alpha\beta\delta\lambda}\, {}_{2}\acute{\gamma}_{\delta\lambda} +$$
$$+ {}_{1}\acute{B}^{\alpha\beta 33}\, {}_{0}\acute{\gamma}_{33} - {}_{1}\acute{T}^{\alpha\beta} \quad , \tag{114b}$$

$$\acute{k}^{\alpha\beta} = {}_{2}\acute{B}^{\alpha\beta\delta\lambda}\, {}_{0}\acute{\gamma}_{\delta\lambda} + {}_{3}\acute{B}^{\alpha\beta\delta\lambda}\, {}_{1}\acute{\gamma}_{\delta\lambda} + {}_{4}\acute{B}^{\alpha\beta\delta\lambda}\, {}_{2}\acute{\gamma}_{\delta\lambda} +$$
$$+ {}_{2}\acute{B}^{\alpha\beta 33}\, {}_{0}\acute{\gamma}_{33} - {}_{2}\acute{T}^{\alpha\beta} \quad , \tag{114c}$$

$$\acute{q}^{\alpha} = 2\left({}_{0}\acute{B}^{\alpha 3\beta 3}\, {}_{0}\acute{\gamma}_{\beta 3} + {}_{0}\acute{B}^{\alpha 3\beta 3}\, {}_{1}\acute{\gamma}_{\beta 3} - {}_{0}\acute{T}^{\alpha 3} \right) \quad , \tag{114d}$$

$$\acute{t}^{\alpha} = 2\left({}_{1}\acute{B}^{\alpha 3\beta 3}\, {}_{0}\acute{\gamma}_{\beta 3} + {}_{2}\acute{B}^{\alpha 3\beta 3}\, {}_{1}\acute{\gamma}_{\beta 3} - {}_{1}\acute{T}^{\alpha 3} \right) \quad , \tag{114e}$$

$$'n^{33} = {}_0'B^{\alpha\beta 33}\,{}_0'\gamma_{\alpha\beta} + {}_1'B^{\alpha\beta 33}\,{}_1'\gamma_{\alpha\beta} + {}_2'B^{\alpha\beta 33}\,{}_2'\gamma_{\alpha\beta} + {}_0'B^{3333}\,{}_0'\gamma_{33} - {}_0'T^{33} \quad , \tag{114f}$$

where

$${}_n'B^{\alpha\beta\delta\lambda} = \int_{'z-'h}^{'z+'h} \mu\, 'C^{\alpha\beta\delta\lambda} (\theta^3)^n \, d\theta^3 \qquad (n = 0,1,2,3,4) \quad , \tag{115a}$$

$${}_n'B^{\alpha\beta 33} = \int_{'z-'h}^{'z+'h} \mu\, 'C^{\alpha\beta 33} (\theta^3)^n \, d\theta^3 \qquad (n = 0,1,2,) \quad , \tag{115b}$$

$${}_n'B^{\alpha 3\beta 3} = \int_{'z-'h}^{'z+'h} \mu\, 'C^{\alpha 3\beta 3} (\theta^3)^n \, d\theta^3 \qquad (n = 0,1,2) \quad , \tag{115c}$$

$${}_0'B^{3333} = \int_{'z-'h}^{'z+'h} \mu\, 'C^{3333} \, d\theta^3 \tag{115d}$$

are defined as isothermal stiffnesses, and

$${}_n'T^{\alpha\beta} = \int_{'z-'h}^{'z+'h} \mu\, 'T \left('C^{\alpha\beta\delta\lambda}\, '\alpha_{\delta\lambda} + 'C^{\alpha\beta 33}\, '\alpha_{33} \right) (\theta^3)^n \, d\theta^3 \quad (n=0,1,2) \quad , \tag{116a}$$

$${}_n'T^{\alpha 3} = \int_{'z-'h}^{'z+'h} \mu\, 'T\, 'C^{\alpha 3\beta 3}\, '\alpha_{\beta 3} (\theta^3)^n \, d\theta^3 \qquad (n=0,1) \quad , \tag{116b}$$

$${}_0'T^{33} = \int_{'z-'h}^{'z+'h} \mu\, 'T \left('C^{\alpha\beta 33}\, '\alpha_{\alpha\beta} + 'C^{3333}\, '\alpha_{33} \right) d\theta^3 \quad , \tag{116c}$$

as thermal stress and couple resultants per unit length of co-ordinate curves on the undeformed reference surface.

Similar expressions can also be obtained for the core and lower facing, respectively.

3.5 Boundary Conditions

The surface integral δJ of (93d) can be split into two parts δJ_1, for the edge boundary surfaces, and δJ_2 for the upper and lower boundary surfaces, so that

$$\delta J = \delta J_1 + \delta J_2 \quad , \tag{117}$$

where

$$\begin{aligned}\delta J_1 = \int_{{}_oC}\int_{-h}^{h} {}_o\nu_\beta \Big\{ \big[\mu \mu^\delta_\alpha \tilde{t}^{\beta\alpha} - \mu\, S^{\alpha\beta}(\mu^\delta_\alpha + V^{*\delta}\|_\alpha - b^\delta_\alpha V^*_3) - \\ - \mu\, S^{\alpha 3}\, V^{*\delta}{}_{,3}\big]\, \delta V^*_\delta + \big[\mu\, \tilde{t}^{\beta 3} - \mu\, S^{\alpha\beta}(V^*_{3,\alpha} + \\ + b^\delta_\alpha V^*_\delta) - \mu\, S^{\alpha 3}(1 + V^*_{3,3})\big]\, \delta V^*_3 \Big\}\, d\theta^3 d{}_os \quad ,\end{aligned} \tag{118a}$$

$$\begin{aligned}\delta J_2 = \int_{{}_oA} {}_on_3 \Big\{ \big[\mu \mu^\delta_\alpha \tilde{t}^{3\alpha} - \mu\, S^{\alpha 3}(\mu^\delta_\alpha + V^{*\delta}\|_\alpha - \\ - b^\delta_\alpha V^*_3) - \mu\, S^{33}\, V^{*\delta}{}_{,3}\big]\, \delta V^*_\delta + \\ + \big[\mu\, \tilde{t}^{33} - \mu\, S^{\alpha 3}(V^*_{3,\alpha} + b^\delta_\alpha V^*_\delta) - \\ - \mu\, S^{33}(1 + V^*_{3,3})\big]\, \delta V^*_3 \Big\}\, dA \quad .\end{aligned} \tag{118b}$$

In a similar manner, we write δJ_1 of (118a) for each layer, substitute Equation (89) into (118a), and integrate across the thickness of the composite shell, to obtain

$$\delta J_1 = \delta 'J_1 + \delta \bar{J}_1 + \delta ''J_1 \quad , \tag{119}$$

where

$$\begin{aligned}
\delta 'J_1 = \int_{{}_0C} \Big[& \Big\{ '\tilde{S}^\lambda - {}_0\nu_\beta ['n^{\beta\delta} (\delta_\delta^\lambda + 'v^\lambda \|_\delta - b_\delta^\lambda 'v_3) + 'm^{\delta\beta} ('\psi^\lambda \|_\delta - \\
& - b_\delta^\lambda - b_\delta^\lambda '\psi_3) + 'q^\beta '\psi^\lambda] \Big\} \delta 'v_\lambda + \Big\{ '\tilde{S}^3 - {}_0\nu_\beta ['n^{\beta\delta} ('v_{3,\delta} + \\
& + b_\delta^\alpha 'v_\alpha) + 'm^{\beta\delta} ('\psi_{3,\delta} + b_\delta^\alpha '\psi_\alpha) + 'q^\beta (1 + '\psi_3)] \Big\} \delta 'v_3 + \\
& + \Big\{ '\tilde{l}^\lambda - {}_0\nu_\beta ['m^{\beta\delta} (\delta_\delta^\lambda + 'v^\lambda \|_\delta - b_\delta^\lambda 'v_3) + 'k^{\beta\delta} ('\psi^\lambda \|_\delta - \\
& - b_\delta^\lambda - b_\delta^\lambda '\psi_3) + 't^\beta '\psi^\lambda] \Big\} \delta '\psi_\lambda + \Big\{ '\tilde{l}^3 - {}_0\nu_\beta ['m^{\beta\delta} ('v_{3,\delta} + \\
& + b_\delta^\alpha 'v_\alpha) + 'k^{\beta\delta} ('\psi_{3,\delta} + b_\delta^\alpha '\psi_\alpha) + \\
& + 't^\beta (1 + '\psi_3)] \Big\} \delta '\psi_3 \Big] \, ds \quad .
\end{aligned} \tag{120}$$

Here

$$\left\{ \begin{matrix} '\tilde{S}^\lambda \\ '\tilde{l}^\lambda \end{matrix} \right\} = \int_{'z-'h}^{'z+'h} \mu \, \mu_\beta^\lambda \, {}_0\nu_\alpha \, \tilde{t}^{\alpha\beta} \left\{ \begin{matrix} 1 \\ \theta^3 \end{matrix} \right\} d\theta^3 \quad , \tag{121a}$$

$$\left\{ \begin{matrix} '\tilde{S}^3 \\ '\tilde{l}^3 \end{matrix} \right\} = \int_{'z-'h}^{'z+'h} \mu \, {}_0\nu_\alpha \, \tilde{t}^{\alpha 3} \left\{ \begin{matrix} 1 \\ \theta^3 \end{matrix} \right\} d\theta^3 \quad . \tag{121b}$$

Expression for $\delta \bar{J}_1$ and $\delta ''J_1$ can be obtained by replacing the prime by a bar and a double prime, respectively.

Substituting Equations (100) into (120), setting Equation (119) equal to zero, and for arbitrary and independent variations

of the displacement components $\delta\bar{u}_i \quad \delta\bar{\psi}_i \quad \delta{}'\psi_i \quad \delta{}''\psi_i$

we obtain

$$
\begin{aligned}
\tilde{S}^\lambda = {} & {}'\tilde{S}^\lambda + \bar{\tilde{S}}^\lambda + {}''\tilde{S}^\lambda \\
= {} & {}_\circ\mu_\beta \Big\{ n^{\beta\delta}(\delta^\lambda_\delta + \bar{u}^\lambda\|_\delta - b^\lambda_\delta \bar{u}_3) + [\bar{m}^{\beta\delta} + \bar{h}({}'n^{\beta\delta} - \\
& - {}''n^{\beta\delta})](\bar{\psi}^\lambda\|_\delta - b^\lambda_\delta \bar{\psi}_3) - m^{\beta\delta} b^\lambda_\delta + ({}'m^{\beta\delta} b^\lambda_\delta) + \\
& + ({}'m^{\beta\delta} - \bar{h}\,{}'n^{\beta\delta})({}'\psi^\lambda\|_\delta - b^\lambda_\delta {}'\psi_3) + ({}''m^{\beta\delta} + \bar{h}\,{}''n^{\beta\delta})({}''\psi^\lambda\|_\delta - \\
& - b^\lambda_\delta {}''\psi_3) + {}'q^\beta\,{}'\psi^\lambda + \bar{q}^\beta \bar{\psi}^\lambda + {}''q^\beta\,{}''\psi^\lambda \Big\} ,
\end{aligned}
\tag{122a}
$$

$$
\begin{aligned}
\tilde{S}^3 = {} & {}'\tilde{S}^3 + \bar{\tilde{S}}^3 + {}''\tilde{S}^3 \\
= {} & {}_\circ\mu_\beta \Big\{ n^{\beta\delta}(\bar{u}_{3,\delta} + b^\alpha_\delta \bar{u}_\alpha) + [\bar{m}^{\beta\delta} + \bar{h}({}'n^{\beta\delta} - {}''n^{\beta\delta})](\bar{\psi}_{3,\delta} + \\
& + b^\alpha_\delta \bar{\psi}_\alpha) + ({}'m^{\beta\delta} - \bar{h}\,{}'n^{\beta\delta})({}'\psi_{3,\delta} + b^\alpha_\delta {}'\psi_\alpha) + ({}''m^{\beta\delta} + \\
& + \bar{h}\,{}''n^{\beta\delta})({}''\psi_{3,\delta} + b^\alpha_\delta {}''\psi_\alpha) + q^\beta + {}'q^\beta\,{}'\psi_3 + \bar{q}^\beta \bar{\psi}_3 + {}''q^\beta\,{}''\psi_3 \Big\} ,
\end{aligned}
\tag{122b}
$$

$$
\begin{aligned}
\tilde{l}^\lambda = {} & \bar{\tilde{l}}^\lambda + \bar{h}({}'\tilde{S}^\lambda - {}''\tilde{S}^\lambda) \\
= {} & {}_\circ\mu_\beta \Big\{ [\bar{m}^{\beta\delta} + \bar{h}({}'n^{\beta\delta} - {}''n^{\beta\delta})](\delta^\lambda_\delta + \bar{u}^\lambda\|_\delta - b^\lambda_\delta \bar{u}_3) + \\
& + [\bar{k}^{\beta\delta} + \bar{h}^2({}'n^{\beta\delta} + {}''n^{\beta\delta})](\bar{\psi}^\lambda\|_\delta - b^\lambda_\delta \bar{\psi}_3) - \\
& - [\bar{k}^{\beta\delta} + \bar{h}({}'m^{\beta\delta} - {}''m^{\beta\delta})] b^\lambda_\delta + \bar{h}({}'m^{\beta\delta} - \\
& - \bar{h}\,{}'n^{\beta\delta})({}'\psi^\lambda\|_\delta - b^\lambda_\delta {}'\psi_3) - ({}''m^{\beta\delta} + \bar{h}\,{}''n^{\beta\delta})\bar{h}({}''\psi^\lambda\|_\delta - \\
& - b^\lambda_\delta {}''\psi_3) + \bar{t}^\beta \bar{\psi}^\lambda + \bar{h}({}'q^\beta\,{}'\psi^\lambda - {}''q^\beta\,{}''\psi^\lambda) \Big\} ,
\end{aligned}
\tag{122c}
$$

$$
\tilde{l}^3 = \bar{\tilde{l}}^3 + \bar{h}({}'\tilde{S}^3 - {}''\tilde{S}^3)
$$

$$= {}_{0}\nu_{\beta} \Big\{ [\bar{m}^{\beta\delta} + \bar{h}({}'n^{\beta\delta} - {}''n^{\beta\delta})](\bar{u}_{3,\delta} + b^{\alpha}_{\delta}\bar{u}_{\alpha}) +$$
$$+ [\bar{k}^{\beta\delta} + \bar{h}^{2}({}'n^{\beta\delta} - {}''n^{\beta\delta})](\bar{\psi}_{3,\delta} + b^{\alpha}_{\delta}\bar{\psi}_{\alpha}) +$$
$$+ \bar{h}({}'m^{\beta\delta} - \bar{h}\,{}'n^{\beta\delta})({}'\psi_{3,\delta} + b^{\alpha}_{\delta}\,{}'\psi_{\alpha}) - \bar{h}({}''m^{\beta\delta} +$$
$$+ \bar{h}\,{}''n^{\beta\delta})({}''\psi_{3,\delta} + b^{\alpha}_{\delta}\,{}''\psi_{\alpha}) + \bar{t}^{\beta}(1 + \bar{\psi}_{3}) +$$
$$+ \bar{h}[{}'q^{\beta}(1 + {}'\psi_{3}) - {}''q^{\beta}(1 + {}''\psi_{3})] \Big\} \quad , \tag{122d}$$

$${}'\tilde{e}^{\lambda} = {}'\tilde{l}^{\lambda} - \bar{h}\,{}'\tilde{S}^{\lambda}$$
$$= {}_{0}\nu_{\beta} \Big\{ ({}'m^{\beta\delta} - \bar{h}\,{}'n^{\beta\delta})[\delta^{\lambda}_{\delta} + \bar{u}^{\lambda}\|_{\delta} + \bar{h}(\bar{\psi}^{\lambda}\|_{\delta} - {}'\psi^{\lambda}\|_{\delta}) -$$
$$- b^{\lambda}_{\delta}\bar{u}_{3} - \bar{h}\,b^{\lambda}_{\delta}(\bar{\psi}_{3} - {}'\psi_{3}) + ({}'k^{\beta\delta} - \bar{h}\,{}'m^{\beta\delta})[{}'\psi^{\lambda}\|_{\delta} -$$
$$- b^{\lambda}_{\delta}(1 + {}'\psi_{3})] + {}'t^{\beta}\,{}'\psi^{\lambda} - \bar{h}\,{}'q^{\beta}\,{}'\psi^{\lambda} \Big\} \quad , \tag{122e}$$

$${}'\tilde{e}^{3} = {}'\tilde{l}^{3} - \bar{h}\,{}'\tilde{S}^{3}$$
$$= {}_{0}\nu_{\beta} \Big\{ ({}'m^{\beta\delta} - \bar{h}\,{}'n^{\beta\delta})[\bar{u}_{3,\delta} + \bar{h}(\bar{\psi}_{3,\delta} - {}'\psi_{3,\delta}) +$$
$$+ b^{\alpha}_{\delta}\bar{u}_{\alpha} + \bar{h}\,b^{\alpha}_{\delta}(\bar{\psi}_{\alpha} - {}'\psi_{\alpha})] + ({}'k^{\beta\delta} - \bar{h}\,{}'m^{\beta\delta})({}'\psi_{3,\delta} +$$
$$+ b^{\alpha}_{\delta}\,{}'\psi_{\alpha}) + {}'t^{\beta}(1 + {}'\psi_{3}) - \bar{h}\,{}'q^{\beta}(1 + {}'\psi_{3}) \Big\} \quad , \tag{122f}$$

$${}''\tilde{e}^{\lambda} = {}''\tilde{l}^{\lambda} - \bar{h}\,{}''\tilde{S}^{\lambda}$$
$$= {}_{0}\nu_{\beta} \Big\{ ({}''m^{\beta\delta} + \bar{h}\,{}''n^{\beta\delta})[\delta^{\lambda}_{\delta} + \bar{u}^{\lambda}\|_{\delta} - \bar{h}(\bar{\psi}^{\lambda}\|_{\delta} -$$
$$- {}''\psi^{\lambda}\|_{\delta}) - b^{\lambda}_{\delta}\bar{u}_{3} + \bar{h}\,b^{\lambda}_{\delta}(\bar{\psi}_{3} - {}''\psi_{3})] + ({}''k^{\beta\delta} +$$
$$+ \bar{h}\,{}''m^{\beta\delta})[{}''\psi^{\lambda}\|_{\delta} - b^{\lambda}_{\delta}(1 + {}''\psi_{3})] + {}''t^{\beta}\,{}''\psi^{\lambda} +$$
$$+ \bar{h}\,{}''q^{\beta}\,{}''\psi^{\lambda} \Big\} \quad , \tag{122g}$$

$$
\begin{aligned}
{}''\tilde{e}^{3} &= {}''\tilde{\ell}^{3} + \bar{h}\,{}''\tilde{s}^{3} \\
&= {}_{0}\gamma_{\beta}\Big\{ ({}''m^{\beta\delta} + \bar{h}\,{}''n^{\beta\delta})\,[\,\bar{u}_{3,\delta} - \bar{h}(\bar{\psi}_{3,\delta} - {}''\psi_{3,\delta}) + b^{\alpha}_{\delta}\bar{u}_{\alpha} - \\
&\quad - \bar{h}\,b^{\alpha}_{\delta}(\bar{\psi}_{\alpha} - {}''\psi_{\alpha})] + ({}''k^{\beta\delta} + \bar{h}\,{}''m^{\beta\delta})({}''\psi_{3,\delta} + b^{\alpha}_{\delta}{}''\psi_{\alpha}) + \\
&\quad + {}''t^{\beta}(1 + {}''\psi_{3}) + \bar{h}\,{}''q^{\beta}(1 + {}''\psi_{3}) \Big\} \qquad . \qquad (122h)
\end{aligned}
$$

B. <u>Boundary conditions at the faces</u>. Evaluating the integrand of Equation (118b) at the upper and lower faces, respectively, and using relations (100), we obtain

$$
\begin{aligned}
\delta J_{2} = \int_{{}_0A} \Big\{ & [(\tilde{p}^{\lambda}_{+} + \tilde{p}^{\lambda}_{-}) - ({}'p^{\lambda}_{+} - {}''p^{\lambda}_{-})]\,\delta\bar{u}_{\lambda} + [(\tilde{p}^{3}_{+} + \\
& + \tilde{p}^{3}_{-}) - ({}'p^{3}_{+} - {}''p^{3}_{-})]\,\delta\bar{u}_{3} + \bar{h}\,[(\tilde{p}^{\lambda}_{+} + \tilde{p}^{\lambda}_{-}) - \\
& - ({}'p^{\lambda}_{+} - {}''p^{\lambda}_{-})]\,\delta\bar{\psi}_{\lambda} + \bar{h}\,[(\tilde{p}^{3}_{+} + \tilde{p}^{3}_{-}) - ({}'p^{3}_{+} - \\
& - {}''p^{3}_{-})]\,\delta\bar{\psi}_{3} + 2\bar{h}\,[(\tilde{p}^{\lambda}_{+} - {}'p^{\lambda}_{+})\,\delta{}'\psi_{\lambda} + \\
& + (\tilde{p}^{3}_{+} - {}'p^{3}_{+})\,\delta{}'\psi_{3}] - 2{}''h\,[(\tilde{p}^{\lambda}_{-} + {}''p^{\lambda}_{-})\,\delta{}''\psi_{\lambda} + \\
& + (\tilde{p}^{3}_{-} + {}''\tilde{p}^{3}_{-})\,\delta{}''\psi_{3}] \Big\}\, dA \qquad , \qquad (123)
\end{aligned}
$$

where

$$
\tilde{p}^{\lambda}_{\pm} = \left[\pm\mu\mu^{\lambda}_{\alpha}\tilde{t}^{3\alpha} \right]_{\theta^{3}=\begin{cases}{}'z+{}'h\\ {}'z-{}'h\end{cases}} \quad , \quad \tilde{p}^{3}_{\pm} = \left[\pm\mu\tilde{t}^{33} \right]_{\theta^{3}=\begin{cases}{}'z+{}'h\\ {}'z-{}'h\end{cases}} \qquad (124)
$$

Setting Equation (123) equal to zero,

$$
\tilde{p}^{i}_{+} = {}'p^{i}_{+} \qquad , \qquad -\tilde{p}^{i}_{-} = {}''p^{i}_{-} \qquad (125)
$$

Here the signs of the prescribed stresses are considered positive when their direction coincides with the positive directions of the normal coordinates.

Up to now all equations are "exact" in the sense that the complete general nonlinear strain-displacement relations have been used. No assumptions about the state of deformation have been made except, of course, that the displacements are linear functions of θ^3. The "exact" problem of composite shells is governed by the equations of motion (105), the stress-displacement relations (109), the constitutive equations (114), and the boundary conditions (122) and (125). These equations represent twelve equations for twelve quantities $\bar{u}_i$, ${}^{\bar{}}\psi_i$, ${}'\psi_i$, ${}''\psi_i$, and, thus form a complete system of equations for the problem. In this form, however, they are, for most practical purposes, too complicated and approximations must be introduced.

CHAPTER IV

SPECIAL APPROXIMATIONS

4.1 General Nonlinear Membrane Theory of Sandwich Shells

Equations (105) can be simplified considerably for sandwich shells with very thin facings of identical thickness, h . For this case

$$ {}'\psi_\alpha = {}'\psi_3 = {}''\psi_\alpha = {}''\psi_3 = 0 \quad , \qquad {}'S^{\alpha 3} = {}''S^{\alpha 3} = 0 \quad , $$

and the last four equations corresponding to the variations $\delta\, {}'\psi_\alpha$, $\delta\, {}'\psi_3$, $\delta\, {}''\psi_\alpha$, and $\delta\, {}''\psi_3$ are dropped. The results follow.

$$ \begin{aligned} \delta\bar{u}_\delta : \quad & \Big\{ n^{\alpha\beta}(\delta^\delta_\beta + \bar{u}^\delta\|_\beta - b^\delta_\beta \bar{u}_3) + [\bar{m}^{\alpha\beta} + ({}'n^{\alpha\beta} - {}''n^{\alpha\beta})\bar{h}]\,(\bar{\psi}^\delta\|_\beta - \\ & - b^\delta_\beta \bar{\psi}_3) - m^{\alpha\beta} b^\delta_\beta + \bar{q}^\alpha \bar{\psi}^\delta \Big\}\|_\alpha - \Big\{ n^{\alpha\beta}(\bar{u}_{3,\beta} + \\ & + b^\lambda_\beta \bar{u}_\lambda) + [\bar{m}^{\alpha\beta} + \bar{h}({}'n^{\alpha\beta} - {}''n^{\alpha\beta})]\,(\bar{\psi}_{3,\beta} + b^\lambda_\beta \bar{\psi}_\lambda) + \\ & + \bar{q}^\alpha \bar{\psi}_3 \Big\} b^\delta_\alpha + p^\delta - f^\delta = 0 \quad . \end{aligned} \tag{126a} $$

$$ \begin{aligned} \delta\bar{u}_3 : \quad & \Big\{ n^{\alpha\beta}(\bar{u}_{3,\beta} + b^\delta_\beta \bar{u}_\delta) + [\bar{m}^{\alpha\beta} + \bar{h}({}'n^{\alpha\beta} - {}''n^{\alpha\beta})]\,(\bar{\psi}_{3,\beta} + \\ & + b^\delta_\beta \bar{\psi}_\delta) + \bar{q}^\alpha(1 + \bar{\psi}_3) \Big\}\|_\alpha + \Big\{ n^{\beta\delta}(\delta^\alpha_\delta + \bar{u}^\alpha\|_\delta - b^\alpha_\delta \bar{u}_3) + \end{aligned} $$

$$+ [\bar{m}^{\beta\delta} + \bar{h}\,({}'n^{\beta\delta} - {}''n^{\beta\delta})]\,(\bar{\Psi}^{\alpha}\|_{\delta} - b^{\alpha}_{\delta}\bar{\Psi}_3) -$$

$$- m^{\beta\delta} b^{\alpha}_{\delta} + \bar{q}^{\beta}\bar{\Psi}^{\alpha} \Big\} b_{\alpha\beta} + p^3 - f^3 = 0 \quad , \qquad (126b)$$

$$\delta\bar{\Psi}_{\delta} : \quad \Big\{ [\bar{m}^{\alpha\beta} + \bar{h}\,({}'n^{\alpha\beta} - {}''n^{\alpha\beta})]\,(\delta^{\delta}_{\beta} + \bar{u}^{\delta}\|_{\beta} - b^{\delta}_{\beta}\bar{u}_3) -$$

$$- [\bar{k}^{\alpha\beta} + \bar{h}\,({}'m^{\alpha\beta} - {}''m^{\alpha\beta})]\, b^{\delta}_{\beta} + [\bar{k}^{\alpha\beta} + \bar{h}^2({}'n^{\alpha\beta} +$$

$$+ {}''n^{\alpha\beta})]\,(\bar{\Psi}^{\delta}\|_{\beta} - b^{\delta}_{\beta}\bar{\Psi}_3) \Big\}\|_{\alpha} + \Big\{ [\bar{m}^{\alpha\beta} +$$

$$+ \bar{h}\,({}'n^{\alpha\beta} - {}''n^{\alpha\beta})]\,(\bar{u}_{3,\beta} + b^{\lambda}_{\beta}\bar{u}_{\lambda}) + [\bar{k}^{\alpha\beta} +$$

$$+ \bar{h}^2({}'n^{\alpha\beta} - {}''n^{\alpha\beta})]\,(\bar{\Psi}_{3,\beta} + b^{\lambda}_{\beta}\bar{\Psi}_{\lambda}) \Big\} b^{\delta}_{\alpha} -$$

$$- \bar{q}^{\alpha}(\delta^{\delta}_{\alpha} + \bar{u}^{\delta}\|_{\alpha} - b^{\delta}_{\alpha}\bar{u}_3) + (\bar{t}^{\alpha}\|_{\alpha} - \bar{n}^{33})\,\bar{\Psi}^{\delta} +$$

$$+ c^{\delta} - m^{\delta} = 0 \quad , \qquad (126c)$$

$$\delta\bar{\Psi}_3 : \quad \Big\{ [\bar{m}^{\alpha\beta} + \bar{h}\,({}'n^{\alpha\beta} - {}''n^{\alpha\beta})]\,(\bar{u}_{3,\beta} + b^{\delta}_{\beta}\bar{u}_{\delta}) +$$

$$+ [\bar{k}^{\alpha\beta} + \bar{h}^2({}'n^{\alpha\beta} + {}''n^{\alpha\beta})]\,(\bar{\Psi}_{3,\beta} + b^{\delta}_{\beta}\bar{\Psi}_{\delta}) \Big\}\|_{\alpha} +$$

$$+ \Big\{ [\bar{m}^{\beta\delta} + \bar{h}\,({}'n^{\beta\delta} - {}''n^{\beta\delta})]\,(\delta^{\alpha}_{\delta} + \bar{u}^{\alpha}\|_{\delta} - b^{\alpha}_{\delta}\bar{u}_3) +$$

$$+ [\bar{k}^{\beta\delta} + \bar{h}^2({}'n^{\beta\delta} + {}''n^{\beta\delta})]\,(\bar{\Psi}^{\alpha}\|_{\delta} - b^{\alpha}_{\delta}\bar{\Psi}_3) - [\bar{k}^{\beta\delta} +$$

$$+ \bar{h}\,({}'m^{\beta\delta} - {}''m^{\beta\delta})]\, b^{\alpha}_{\delta} \Big\} b_{\alpha\beta} - \bar{q}^{\alpha}(\bar{u}_{3,\alpha} +$$

$$+ b^{\beta}_{\alpha}\bar{u}_{\beta}) + (\bar{t}^{\alpha}\|_{\alpha} - \bar{n}^{33})\,(1 + \bar{\Psi}_3) + c^3 - m^3 = 0 \ . \qquad (126d)$$

These are six equations corresponding to the six displacement functions $\bar{u}_i$ and $\bar{\Psi}_i$.

The related boundary conditions can be derived from (122) and (125) as follows

$$\tilde{S}^{\lambda} = {}_{0}\nu_{\beta} \Big\{ n^{\beta\delta} (\delta^{\lambda}_{\delta} + \bar{u}^{\lambda}\|_{\delta} - b^{\lambda}_{\delta} \bar{u}_3) + [\bar{m}^{\beta\delta} + \bar{h} ({}'n^{\beta\delta} - {}''n^{\beta\delta})] (\bar{\Psi}^{\lambda}\|_{\delta} - b^{\lambda}_{\delta} \bar{\Psi}_3) - m^{\beta\delta} b^{\lambda}_{\delta} + \bar{q}^{\beta} \bar{\Psi}^{\lambda} \Big\} , \tag{127a}$$

$$\tilde{S}^{3} = {}_{0}\nu_{\beta} \Big\{ n^{\beta\delta} (\bar{u}_{3,\delta} + b^{\alpha}_{\delta} \bar{u}_{\alpha}) + [\bar{m}^{\beta\delta} + \bar{h} ({}'n^{\beta\delta} - {}''n^{\beta\delta})] (\bar{\Psi}_{3,\delta} + b^{\alpha}_{\delta} \bar{\Psi}_{\alpha}) + \bar{q}^{\beta} + \bar{q}^{\beta} \bar{\Psi}_3 \Big\} , \tag{127b}$$

$$\tilde{t}^{\lambda} = {}_{0}\nu_{\beta} \Big\{ [\bar{m}^{\beta\delta} + \bar{h} ({}'n^{\beta\delta} - {}''n^{\beta\delta})] (\delta^{\lambda}_{\delta} + \bar{u}^{\lambda}\|_{\delta} - b^{\lambda}_{\delta} \bar{u}_3) + [\bar{k}^{\beta\delta} + \bar{h}^2 ({}'n^{\beta\delta} + {}''n^{\beta\delta})] (\bar{\Psi}^{\lambda}\|_{\delta} - b^{\lambda}_{\delta} \bar{\Psi}_3) + [\bar{k}^{\beta\delta} + \bar{h} ({}'m^{\beta\delta} - {}''m^{\beta\delta})] b^{\lambda}_{\delta} + \bar{t}^{\beta} \bar{\Psi}^{\lambda} \Big\} , \tag{127c}$$

$$\tilde{t}^{3} = {}_{0}\nu_{\beta} \Big\{ [\bar{m}^{\beta\delta} + \bar{h} ({}'n^{\beta\delta} - {}''n^{\beta\delta})] (\bar{u}_{3,\delta} + b^{\alpha}_{\delta} \bar{u}_{\alpha}) + [\bar{k}^{\beta\delta} + \bar{h}^2 ({}'n^{\beta\delta} + {}''n^{\beta\delta})] (\bar{\Psi}_{3,\delta} + b^{\alpha}_{\delta} \bar{\Psi}_{\alpha}) + \bar{t}^{\beta} (1 + \bar{\Psi}_3) \Big\} , \tag{127d}$$

and

$$\tilde{p}^{i}_{+} = \bar{p}^{i}_{+} , \qquad -\tilde{p}^{i}_{-} = \bar{p}^{i}_{-} \tag{127e}$$

A further simplification of (126) and (127) is possible for the case when $\bar{\Psi}_3 = 0$, so that one more equation, corresponding to the variation $\delta \bar{\Psi}_3$, should be suppressed in (105). The equations of motion then are

$$\delta \bar{u}_\delta : \quad \Big\{ n^{\alpha\beta} (\delta^\delta_\beta + \bar{u}^\delta\|_\beta - b^\delta_\beta \bar{u}_3) + [\bar{m}^{\alpha\beta} + \bar{h} ({}'n^{\alpha\beta} - {}''n^{\alpha\beta})] \bar{\psi}^\delta\|_\beta - m^{\alpha\beta} b^\delta_\beta + \bar{q}^\alpha \bar{\psi}^\delta \Big\}\|_\alpha - \Big\{ n^{\alpha\beta} (\bar{u}_{3,\beta} + b^\lambda_\beta \bar{u}_\lambda) + [\bar{m}^{\alpha\beta} + \bar{h} ({}'n^{\alpha\beta} - {}''n^{\alpha\beta})] b^\lambda_\beta \bar{\psi}_\lambda \Big\} b^\delta_\alpha + p^\delta - f^\delta = 0 , \tag{128a}$$

$$\delta \bar{u}_3 : \quad \Big\{ n^{\alpha\beta} (\bar{u}_{3,\beta} + b^\delta_\beta \bar{u}_\delta) + [\bar{m}^{\alpha\beta} + \bar{h} ({}'n^{\alpha\beta} - {}''n^{\alpha\beta})] b^\delta_\beta \bar{\psi}_\delta + \bar{q}^\alpha \Big\}\|_\alpha + \Big\{ n^{\beta\delta} (\delta^\alpha_\delta + \bar{u}^\alpha\|_\delta - b^\alpha_\delta \bar{u}_3) + [\bar{m}^{\beta\delta} + \bar{h} ({}'n^{\beta\delta} - {}''n^{\beta\delta})] \bar{\psi}^\alpha\|_\delta - m^{\beta\delta} b^\alpha_\delta + \bar{q}^\beta \bar{\psi}^\alpha \Big\} b_{\alpha\beta} + p^3 - f^3 = 0 , \tag{128b}$$

$$\delta \bar{\psi}_\delta : \quad \Big\{ [\bar{m}^{\alpha\beta} + \bar{h} ({}'n^{\alpha\beta} - {}''n^{\alpha\beta})] (\delta^\delta_\beta + \bar{u}^\delta\|_\beta - b^\delta_\beta \bar{u}_3) - [\bar{k}^{\alpha\beta} + \bar{h} ({}'m^{\alpha\beta} - {}''m^{\alpha\beta})] b^\delta_\beta + [\bar{k}^{\alpha\beta} + \bar{h}^2 ({}'n^{\alpha\beta} + {}''n^{\alpha\beta})] \times \bar{\psi}^\delta\|_\beta \Big\}\|_\alpha + \Big\{ [\bar{m}^{\alpha\beta} + \bar{h} ({}'n^{\alpha\beta} - {}''n^{\alpha\beta})] (\bar{u}_{3,\beta} + b^\lambda_\beta \bar{u}_\lambda) + [\bar{k}^{\alpha\beta} + \bar{h}^2 ({}'n^{\alpha\beta} + {}''n^{\alpha\beta})] b^\lambda_\beta \bar{\psi}_\lambda \Big\} b^\delta_\alpha - \bar{q}^\alpha (\delta^\delta_\alpha + \bar{u}^\delta\|_\alpha - b^\delta_\alpha \bar{u}_3) + (\bar{t}^\alpha\|_\alpha - \bar{n}^{33}) \bar{\psi}^\delta + c^\delta - m^\delta = 0 , \tag{128c}$$

and the boundary conditions are

$$\tilde{S}^\lambda = \nu_\beta \Big\{ n^{\beta\delta} (\delta^\lambda_\delta + \bar{u}^\lambda_\delta - b^\lambda_\delta \bar{u}_3) + [\bar{m}^{\beta\delta} + \bar{h} ({}'n^{\beta\delta} - {}''n^{\beta\delta})] \bar{\psi}^\lambda\|_\delta - m^{\beta\delta} b^\lambda_\delta + \bar{q}^\beta \bar{\psi}^\lambda \Big\} , \tag{129a}$$

$$\tilde{S}^{3} = {}_{0}\gamma_{\beta} \{ n^{\beta\delta} (\bar{u}_{3,\delta} + b^{\alpha}_{\delta} \bar{u}_{\alpha}) + [\bar{m}^{\beta\delta} + \bar{h} ({}'n^{\beta\delta} - {}''n^{\beta\delta})] b^{\alpha}_{\delta} \bar{\Psi}_{\alpha} + \bar{q}^{\beta} \} \quad , \tag{129b}$$

$$\tilde{t}^{\lambda} = {}_{0}\gamma_{\beta} \{ [\bar{m}^{\beta\delta} + \bar{h} ({}'n^{\beta\delta} - {}''n^{\beta\delta})] (\delta^{\lambda}_{\delta} + \bar{u}^{\lambda} \|_{\delta} - b^{\lambda}_{\delta} \bar{u}_{3}) + [\bar{k}^{\beta\delta} + \bar{h}^{2} ({}'n^{\beta\delta} + {}''n^{\beta\delta})] \bar{\Psi}^{\lambda} \|_{\delta} + [\bar{k}^{\beta\delta} + \bar{h} ({}'m^{\beta\delta} - {}''m^{\beta\delta})] b^{\lambda}_{\delta} + \bar{t}^{\beta} \bar{\Psi}^{\lambda} \} \; , \tag{129c}$$

and

$$\tilde{P}^{i}_{+} = \bar{P}^{i}_{+} \quad , \qquad - \tilde{P}^{i}_{-} = \bar{P}^{i}_{-} \quad , \tag{129d}$$

where the definitions of $\bar{P}^{i}_{+}$, $\bar{P}^{i}_{-}$ are also simplified accordingly.

4.2 Partially Nonlinear Theory of Sandwich Shells

The modified Hellinger-Reissner variational theorem of Chapter II is now used in the formulation of a simplified nonlinear theory of sandwich shells on the basis of the following partially nonlinear version of the strain-displacement relations (39)

$$\gamma_{ij} = \frac{1}{2} (V_{i}|_{j} + V_{j}|_{i} + V^{3}|_{i} V_{3}|_{j}) \quad . \tag{130}$$

These relations had been used by Ebcioglu in (12) and (23) for deriving nonlinear theories of plates and shells, respectively.

Using (130), the variational equation (86) becomes

$$\delta \left\{ \int_{{}_{0}\tau} [s^{ij} \gamma_{ij} - \Phi] \, d\tau - \int_{{}_{0}\tau} \frac{1}{2} s^{ij} (V_{i}|_{j} + V_{j}|_{i} + V^{3}|_{i} V_{3}|_{j}) \, d\tau + \int_{{}_{0}\tau} \rho_{0} ({}_{0}b^{i} - {}_{0}a^{i}) V_{i} \, d\tau + \int_{{}_{0}S} {}_{0}n_{j} \tilde{t}^{ji} V_{i} \, dS \right\} = 0 \quad . \tag{131}$$

With the help of Equations (83) - (83), we can shift the components of displacement into the reference surface, so that

$$
\begin{aligned}
\delta\Big[\int_{{}_0\tau} [\, S^{ij}\gamma_{ij} - \Phi \,]\, d\tau \; - \int_{{}_0\tau} \tfrac{1}{2} S^{\alpha\beta}[(\mu^{\delta}_{\alpha})(V^*_{\delta\|\beta} - b_{\delta\beta}V^*_3) \; + \\
+ \mu^{\delta}_{\beta}(V^*_{\delta\|\alpha} - b_{\delta\alpha}V^*_3) + (V^*_{3,\alpha} + b^{\delta}_{\alpha}V^*_{\delta})(V^*_{3,\beta} + b^{\lambda}_{\beta}V^*_{\lambda}) \; + \\
+ S^{\alpha 3}[\mu^{\delta}_{\alpha}V^*_{\delta,3} + V^*_{3,\alpha} + b^{\delta}_{\alpha}V^*_{\delta} + (V^*_{3,\alpha} + b^{\delta}_{\alpha}V^*_{\delta})\, V^*_{3,3}\,] \; + \\
+ \tfrac{1}{2} S^{33}[2V^*_{3,3} + V^*_{3,3}\, V^*_{3,3}]\Big\}\, d\tau \; + \int_{{}_0\tau}[\rho_0({}_0b^{\alpha} - {}_0a^{\alpha})\mu^{\delta}_{\alpha}V^*_{\delta} \; + \\
+ \rho_0({}_0b^3 - {}_0a^3)\, V^*_3\,]\, d\tau \; + \int_{{}_0S}[({}_0n_{\beta}\tilde{t}^{\beta\alpha} + {}_0n_3\tilde{t}^{3\alpha})\,\mu^{\delta}_{\alpha}V^*_{\delta} \; + \\
+ ({}_0n_{\beta}\tilde{t}^{\beta 3} + {}_0n_3\tilde{t}^{33})\, V^*_3\,]\; dS\Big] = 0 \quad .
\end{aligned}
\tag{132}
$$

Executing the variation and using Green's transformation, we obtain

$$
\delta I_1 + \delta I_2 + \delta I_3 + \delta J = 0 \qquad , \tag{133}
$$

where

$$
\begin{aligned}
\delta I_1 = \int_{{}_0\tau} \Big\{ \big\{ (\mu\mu^{\delta}_{\alpha}S^{\alpha\beta})_{\|\beta} - [\mu S^{\alpha\beta}(V^*_{3,\alpha} + b^{\lambda}_{\alpha}V^*_{\lambda}) + \mu S^{\beta 3}(1 \; + \\
+ V^*_{3,3})]\, b^{\delta}_{\beta} + (\mu\mu^{\delta}_{\alpha}S^{\alpha 3})_{,3} + \mu\mu^{\delta}_{\alpha}\rho_0({}_0b^{\alpha} - {}_0a^{\alpha})\big\}\,\delta V^*_{\delta} \; + \\
+ \big\{ (\mu\mu^{\delta}_{\alpha}S^{\alpha\beta})\, b_{\delta\beta} + [\mu S^{\alpha\beta}(V^*_{3,\alpha} + b^{\delta}_{\alpha}V^*_{\delta}) + \mu S^{\beta 3}(1 \; + \\
+ V^*_{3,3})]_{\|\beta} + [\mu S^{\alpha 3}(V^*_{3,\alpha} + b^{\delta}_{\alpha}V^*_{\delta}) + \mu S^{33}(1 \; + \\
+ V^*_{3,3})]_{,3} + \mu\rho_0({}_0b^3 - {}_0a^3)\,\delta V^*_3 \Big\}\, d\theta^3 dA \qquad ,
\end{aligned}
\tag{134a}
$$

$$
\begin{aligned}
\delta I_2 = \int_{{}_0\tau} \Big\{ \big\{ \gamma_{\alpha\beta} - \tfrac{1}{2}[\mu^{\delta}_{\alpha}(V^*_{\delta\|\beta} - b_{\delta\beta}V^*_3) + \mu^{\delta}_{\beta}(V^*_{\delta\|\alpha} - b_{\delta\alpha}V^*_3) \; + \\
+ (V^*_{3,\alpha} + b^{\delta}_{\alpha}V^*_{\delta})(V^*_{3,\beta} + b^{\delta}_{\beta}V^*_{\delta})\,]\big\}\, \delta S^{\alpha\beta} \; +
\end{aligned}
$$

$$+\Big\{2\gamma_{\alpha 3}-\big[\mu^{\beta}_{\alpha}V^{*}_{\beta,3}+V^{*}_{3,\alpha}+b^{\delta}_{\alpha}V^{*}_{\delta}+(V^{*}_{3,\alpha}+$$

$$+b^{\delta}_{\alpha}V^{*}_{\delta})V^{*}_{3,3}\big]\Big\}\delta S^{\alpha 3}+\Big\{\gamma_{33}-\tfrac{1}{2}\big[2V^{*}_{3,3}+$$

$$+V^{*}_{3,3}V^{*}_{3,3}\big]\Big\}\delta S^{33}\Big]\mu\, dA\, d\theta^{3} \quad , \tag{134b}$$

$$\delta I_3=\int_{{}_0\tau}\Big[S^{ij}-\frac{1}{2}\Big(\frac{\partial\Phi}{\partial\gamma_{ij}}+\frac{\partial\Phi}{\partial\gamma_{ji}}\Big)\Big]\delta\gamma_{ij}\,\mu\, dA\, d\theta^{3} \quad , \tag{134c}$$

$$\delta J=\int_{{}_0C}\int_{-h}^{h}\Big\{{}_0\nu_{\beta}\big[\mu\mu^{\delta}_{\alpha}\tilde{t}^{\beta\alpha}-\mu\mu^{\delta}_{\alpha}S^{\alpha\beta}\big]\delta V^{*}_{\delta}+{}_0\nu_{\beta}\big[\mu\tilde{t}^{\beta 3}-\mu S^{\alpha\beta}(V^{*}_{3,\alpha}+$$

$$+b^{\delta}_{\alpha}V^{*}_{\delta})-\mu S^{\alpha 3}(1+V^{*}_{3,3})\big]\delta V^{*}_{3}\Big\}d\theta^{3}\, ds +$$

$$+\int_{{}_0A}\Big\{{}_0n_3(\mu\mu^{\delta}_{\alpha}\tilde{t}^{3\alpha}-\mu\mu^{\delta}_{\alpha}S^{\alpha 3})\delta V^{*}_{\delta}+{}_0n_3\big[\mu\tilde{t}^{33}-$$

$$-\mu S^{\alpha 3}(V^{*}_{3,\alpha}+b^{\delta}_{\alpha}V^{*}_{\delta})-\mu S^{33}(1+$$

$$+V^{*}_{3,3})\big]\delta V^{*}_{3}\Big\}\, dA \quad , \tag{134d}$$

In a manner similar to that of the previous chapter we obtain the field equations of a partially nonlinear theory of composite shells as follows.

The equations of motion are

$$\delta\bar{u}_{\delta}:\ (n^{\alpha\delta}-m^{\alpha\beta}b^{\delta}_{\beta})\|_{\alpha}-\Big\{n^{\alpha\beta}(\bar{u}_{3,\beta}+b^{\lambda}_{\beta}\bar{u}_{\lambda})+$$

$$+\big[\bar{m}^{\alpha\beta}+\bar{h}({}'n^{\alpha\beta}-{}''n^{\alpha\beta})\big](\bar{\psi}_{3,\beta}+b^{\lambda}_{\beta}\bar{\psi}_{\lambda})+$$

$$+({}'m^{\alpha\beta}-\bar{h}\,{}'n^{\alpha\beta})({}'\psi_{3,\beta}+b^{\lambda}_{\beta}\,{}'\psi_{\lambda})+({}''m^{\alpha\beta}+\bar{h}\,{}''n^{\alpha\beta})({}''\psi_{3,\beta}+$$

$$+b^{\lambda}_{\beta}\,{}''\psi_{\lambda})+q^{\alpha}+{}'q^{\alpha}\,{}'\psi_3+\bar{q}^{\alpha}\bar{\psi}_3+{}''q^{\alpha}\,{}''\psi_3\Big\}b^{\delta}_{\alpha}+$$

$$+p^{\delta}-f^{\delta}=0 \quad , \tag{135a}$$

$$\begin{aligned}
\delta \bar u_3 :\quad & \Big\{ n^{\alpha\beta}(\bar u_{3,\beta} + b^\delta_\beta \bar u_\delta) + [\bar m^{\alpha\beta} + \bar h({}'n^{\alpha\beta} - {}''n^{\alpha\beta})](\bar\psi_{3,\beta} + \\
& + b^\delta_\beta \bar\psi_\delta) + ({}'m^{\alpha\beta} - \bar h\,{}'n^{\alpha\beta})({}'\psi_{3,\beta} + b^\delta_\beta\,{}'\psi_\delta) + \\
& + ({}''m^{\alpha\beta} + \bar h\,{}''n^{\alpha\beta})({}''\psi_{3,\beta} + b^\delta_\beta\,{}''\psi_\delta) + q^\alpha + {}'q^\alpha\,{}'\psi_3 + \\
& + \bar q^\alpha \bar\psi_3 + {}''q^\alpha\,{}''\psi_3 \Big\}\Big\|_\alpha + (n^{\alpha\beta} - m^{\alpha\delta} b^\beta_\delta)\, b_{\alpha\beta} + \\
& + p^3 - f^3 = 0 \quad ,
\end{aligned} \tag{135b}$$

$$\begin{aligned}
\delta \bar\psi_\delta :\quad & \Big\{[\bar m^{\alpha\delta} + \bar h({}'n^{\alpha\delta} - {}''n^{\alpha\delta})] - [\bar k^{\alpha\beta} + \bar h({}'m^{\alpha\beta} - {}''m^{\alpha\beta})]\, b^\delta_\beta \Big\}\Big\|_\alpha - \\
& - \Big\{[\bar m^{\alpha\beta} + \bar h({}'n^{\alpha\beta} - {}''n^{\alpha\beta})](\bar u_{3,\beta} + b^\lambda_\beta \bar u_\lambda) + [\bar k^{\alpha\beta} + \bar h^2({}'n^{\alpha\beta} + \\
& + {}''n^{\alpha\beta})](\bar\psi_{3,\beta} + b^\lambda_\beta \bar\psi_\lambda) + \bar h({}'m^{\alpha\beta} - \bar h\,{}'n^{\alpha\beta})({}'\psi_{3,\beta} + b^\lambda_\beta\,{}'\psi_\lambda) - \\
& - \bar h({}''m^{\alpha\beta} + \bar h\,{}''n^{\alpha\beta})({}''\psi_{3,\beta} + b^\lambda_\beta\,{}''\psi_\lambda) + [{}'q^\alpha(1 + {}'\psi_3) - {}''q^\alpha(1 + \\
& + {}''\psi_3)]\,\bar h \Big\}\, b^\delta_\alpha - \bar q^\delta + c^\delta - m^\delta = 0 \quad ,
\end{aligned} \tag{135c}$$

$$\begin{aligned}
\delta \bar\psi_3 :\quad & \Big\{[\bar m^{\alpha\beta} + \bar h({}'n^{\alpha\beta} - {}''n^{\alpha\beta})](\bar u_{3,\beta} + b^\delta_\beta \bar u_\delta) + [\bar k^{\alpha\beta} + \bar h^2({}'n^{\alpha\beta} + \\
& + {}''n^{\alpha\beta})](\bar\psi_{3,\beta} + b^\delta_\beta \bar\psi_\delta) + \bar h({}'m^{\alpha\beta} - \bar h\,{}'n^{\alpha\beta})({}'\psi_{3,\beta} + \\
& + b^\delta_\beta\,{}'\psi_\delta) - \bar h({}''m^{\alpha\beta} + \bar h\,{}''n^{\alpha\beta})({}''\psi_{3,\beta} + b^\delta_\beta\,{}''\psi_\delta) + \\
& + [{}'q^\alpha(1 + {}'\psi_3) - {}''q^\alpha(1 + {}''\psi_3)]\,\bar h \Big\}\Big\|_\alpha + \Big\{[\bar m^{\beta\delta} + \bar h({}'n^{\beta\delta} - \\
& - {}''n^{\beta\delta})] + [\bar k^{\beta\alpha} + \bar h({}'m^{\beta\alpha} - {}''m^{\beta\alpha})]\, b^\delta_\alpha \Big\}\, b_{\beta\delta} - \bar q^\alpha(\bar u_{3,\alpha} + \\
& + b^\beta_\alpha \bar u_\beta) + (\bar t^\alpha\|_\alpha - \bar n^{33})(1 + \bar\psi_3) + c^3 - m^3 = 0 \quad ,
\end{aligned} \tag{135d}$$

$$\delta {}'\psi_\delta: \quad [({}'m^{\alpha\delta} - \bar{h}\,{}'n^{\alpha\delta}) - ({}'k^{\alpha\beta} - \bar{h}\,{}'m^{\alpha\beta})\, b^\delta_\beta]\|_\alpha - \{ ({}'m^{\alpha\beta} - \bar{h}\,{}'n^{\alpha\beta}) [\bar{u}_{3,\beta} + \bar{h}(\bar{\psi}_{3,\beta} - {}'\psi_{3,\beta}) + b^\lambda_\beta \bar{u}_\lambda + \bar{h}\, b^\lambda_\beta (\bar{\psi}_\lambda - {}'\psi_\lambda)] - ({}'k^{\alpha\beta} - \bar{h}\,{}'m^{\alpha\beta})({}'\psi_{3,\beta} + b^\lambda_\beta\, {}'\psi_\lambda) - {}'q^\alpha \bar{h}(1 + {}'\psi_3) \}\, b^\delta_\alpha - {}'q^\delta + {}'g^\delta - {}'d^\delta = 0 \quad , \tag{135e}$$

$$\delta {}'\psi_3: \quad \{ ({}'m^{\alpha\beta} - \bar{h}\,{}'n^{\alpha\beta}) [\bar{u}_{3,\beta} + \bar{h}(\bar{\psi}_{3,\beta} - {}'\psi_{3,\beta}) + b^\delta_\beta \bar{u}_\delta + \bar{h}\, b^\delta_\beta (\bar{\psi}_\delta - {}'\psi_\delta)] + ({}'k^{\alpha\beta} - \bar{h}\,{}'m^{\alpha\beta})({}'\psi_{3,\beta} + b^\delta_\beta\, {}'\psi_\delta) - {}'q^\delta \bar{h}(1 + {}'\psi_3) \}\|_\alpha + [({}'m^{\beta\alpha} - \bar{h}\,{}'n^{\beta\alpha}) - ({}'k^{\beta\delta} - \bar{h}\,{}'m^{\beta\delta})\, b^\alpha_\delta]\, b_{\alpha\beta} - {}'q^\alpha [\bar{u}_{3,\alpha} + \bar{h}(\bar{\psi}_{3,\alpha} - {}'\psi_{3,\alpha}) + b^\beta_\alpha \bar{u}_\beta + \bar{h}\, b^\beta_\alpha (\bar{\psi}_\beta - {}'\psi_\beta)] + ({}'t^\alpha\|_\alpha - {}'n^{33})(1 + {}'\psi_3) + {}'g^3 - {}'d^3 = 0 \quad , \tag{135f}$$

$$\delta {}''\psi_\delta: \quad [({}''m^{\alpha\delta} + \bar{h}\,{}''n^{\alpha\delta}) - ({}''k^{\alpha\beta} + \bar{h}\,{}''m^{\alpha\beta})\, b^\delta_\beta]\|_\alpha - \{ ({}''m^{\alpha\beta} + \bar{h}\,{}''n^{\alpha\beta}) [\bar{u}_{3,\beta} - \bar{h}(\bar{\psi}_{3,\beta} - {}''\psi_{3,\beta}) + b^\lambda_\beta \bar{u}_\lambda - \bar{h}\, b^\lambda_\beta (\bar{\psi}_\lambda - {}''\psi_\lambda)] + ({}''k^{\alpha\beta} + \bar{h}\,{}''m^{\alpha\beta})({}''\psi_{3,\beta} + b^\lambda_\beta\, {}''\psi_\lambda) + \bar{h}\,{}''q^\alpha (1 + {}''\psi_3) \}\, b^\delta_\alpha - {}''q^\delta + {}''g^\delta + {}''d^\delta = 0 \quad , \tag{135g}$$

$$\delta {}''\psi_3: \quad \{ ({}''m^{\alpha\beta} + \bar{h}\,{}''n^{\alpha\beta}) [\bar{u}_{3,\beta} - \bar{h}(\bar{\psi}_{3,\beta} - {}''\psi_{3,\beta}) + b^\delta_\beta \bar{u}_\delta - \bar{h}\, b^\delta_\beta (\bar{\psi}_\delta - {}''\psi_\delta)] + ({}''k^{\alpha\beta} + \bar{h}\,{}''m^{\alpha\beta})({}''\psi_{3,\beta} + b^\delta_\beta\, {}''\psi_\delta) + {}''q^\alpha \bar{h}(1 + {}''\psi_3) \}\|_\alpha + [({}''m^{\alpha\beta} + \bar{h}\,{}''n^{\alpha\beta}) - ({}''k^{\beta\delta} + \bar{h}\,{}''m^{\beta\delta})\, b^\alpha_\delta]\, b_{\alpha\beta} - {}''q^\alpha [\bar{u}_{3,\alpha} - \bar{h}(\bar{\psi}_{3,\alpha} - {}''\psi_{3,\alpha}) + b^\beta_\alpha \bar{u}_\beta - \bar{h}\, b^\beta_\alpha (\bar{\psi}_\beta - {}''\psi_\beta)] + ({}''t^\alpha\|_\alpha - {}''n^{33})(1 + {}''\psi_3) + {}''g^3 + {}''d^3 = 0 \quad , \tag{135h}$$

where

$$\begin{Bmatrix} {}'p^\alpha_\pm \\ {}'C^\alpha_\pm \end{Bmatrix} = \left[(\mu\, \mu^\alpha_\beta\, {}'s^{3\beta}) \begin{Bmatrix} 1 \\ \theta^3 \end{Bmatrix} \right] \theta^3 = {}'z \pm {}'h \quad , \tag{136a}$$

$$\left\{ \begin{matrix} {}'p^3_{\pm} \\ {}'c^3_{\pm} \end{matrix} \right\} = \left[\left(\mu\, {}'S^{3\alpha} [\, {}'v_{3,\alpha} + b^{\beta}_{\alpha}\, {}'v_{\beta} + \theta^3 (\, {}'\psi_{3,\alpha} + b^{\beta}_{\alpha}\, {}'\psi_{\beta})] + \mu\, {}'S^{33} (1 + {}'\psi_3) \right) \left\{ \begin{matrix} 1 \\ \theta^3 \end{matrix} \right\} \right]_{\theta^3 = {}'z \pm {}'h} \quad (136b)$$

and the definition for ${}'f^i$ and ${}'m^i$ are based on Equation (104). Similar definitions for the core and lower facing can be obtained by replacing the prime by a bar and a double prime, respectively.

The strain-displacement relations for the upper facing are

$$ {}'_{0}\gamma_{\alpha\beta} = \frac{1}{2} [\, \mu^{\delta}_{\alpha} (\, {}'v_{\delta \| \beta} - b_{\delta\beta}\, {}'v_3) + \mu^{\delta}_{\beta} (\, {}'v_{\delta \| \alpha} - b_{\delta\alpha}\, {}'v_3) + (\, {}'v_{3,\alpha} + b^{\delta}_{\alpha}\, {}'v_{\delta})(\, {}'v_{3,\beta} + b^{\lambda}_{\beta}\, {}'v_{\lambda})] \quad , \quad (137a)$$

$$ {}'_{1}\gamma_{\alpha\beta} = \frac{1}{2} [\, \mu^{\delta}_{\alpha} (\, {}'\psi_{\delta \| \beta} - b_{\delta\beta}\, {}'\psi_3) + \mu^{\delta}_{\beta} (\, {}'\psi_{\delta \| \alpha} - b_{\delta\alpha}\, {}'\psi_3) + (\, {}'v_{3,\alpha} + b^{\delta}_{\alpha}\, {}'v_3)(\, {}'\psi_{3,\beta} + b^{\lambda}_{\beta}\, {}'\psi_{\lambda}) + (\, {}'\psi_{3,\alpha} + b^{\delta}_{\alpha}\, {}'\psi_{\delta})(\, {}'v_{3,\beta} + b^{\lambda}_{\beta}\, {}'v_{\lambda})] \quad , \quad (137b)$$

$$ {}'_{2}\gamma_{\alpha\beta} = \frac{1}{2} [(\, {}'\psi_{3,\alpha} + b^{\delta}_{\alpha}\, {}'\psi_{\delta})(\, {}'\psi_{3,\beta} + b^{\nu}_{\beta}\, {}'\psi_{\nu})] \quad , \quad (137c)$$

$$ {}'_{0}\gamma_{\alpha 3} = \frac{1}{2} [(\, {}'v_{3,\alpha} + b^{\delta}_{\alpha}\, {}'v_{\delta})(1 + {}'\psi_3) + {}'\psi_{\alpha}] \quad , \quad (137d)$$

$$ {}'_{1}\gamma_{\alpha 3} = \frac{1}{2}\, {}'\psi_{3,\beta} (1 + {}'\psi_3) \quad , \quad (137e)$$

$$ {}'_{0}\gamma_{33} = \frac{1}{2} [2\, {}'\psi_3 + (\, {}'\psi_3)^2] \quad . \quad (137f)$$

Similar relations for the core and lower facing can also be obtained.

The general constitutive equations for ${}'n^{\alpha\beta}$, ${}'m^{\alpha\beta}$, ${}'q^{\alpha}$, etc. are similar to (114), keeping in mind the simpler functional

relationship, expressed by (137), between the components of strain and displacement.

Finally, the boundary conditions for this case are to prescribe

$$\tilde{p}^{i}_{+} = {}'p^{i}_{+} \qquad , \qquad -\tilde{p}^{i}_{-} = {}''p^{i}_{-}$$

and

$$\tilde{S}^{\lambda} = {}_{0}\nu_{\beta}\,(n^{\beta\lambda} - b^{\lambda}_{\delta}\, m^{\beta\delta}) \qquad , \tag{138a}$$

$$\begin{aligned}\tilde{S}^{3} = {}_{0}\nu_{\beta} \Big\{ & n^{\beta\delta}(\bar{u}_{3,\delta} + b^{\alpha}_{\delta}\bar{u}_{\alpha}) + [\bar{m}^{\beta\delta} + \bar{h}\,({}'n^{\beta\delta} - {}''n^{\beta\delta})](\bar{\psi}_{3,\delta} + b^{\alpha}_{\delta}\bar{\psi}_{\alpha}) + \\ & + ({}'m^{\beta\delta} - \bar{h}\,{}'n^{\beta\delta})({}'\psi_{3,\delta} + b^{\alpha}_{\delta}\,{}'\psi_{\alpha}) + ({}''m^{\beta\delta} + \bar{h}\,{}''n^{\beta\delta})({}''\psi_{3,\delta} + \\ & + b^{\alpha}_{\delta}\,{}''\psi_{\alpha}) + {}'q^{\beta} + {}'q^{\beta}\,{}'\psi_{3} + {}''q^{\beta}\,{}''\psi_{3} + \bar{q}^{\beta}\bar{\psi}_{3} \Big\} \quad ,\end{aligned} \tag{138b}$$

$$\tilde{l}^{\lambda} = {}_{0}\nu_{\beta}\Big\{[\bar{m}^{\beta\lambda} + \bar{h}\,({}'n^{\beta\lambda} - {}''n^{\beta\lambda})] - [\bar{k}^{\beta\delta} + \bar{h}\,({}''m^{\beta\delta} - {}''n^{\beta\delta})]\, b^{\lambda}_{\delta}\Big\} \qquad , \tag{138c}$$

$$\begin{aligned}\tilde{l}^{3} = {}_{0}\nu_{\beta}\Big\{ & [\bar{m}^{\beta\lambda} + \bar{h}\,({}'n^{\beta\lambda} - {}''n^{\beta\lambda})]\,(\bar{u}_{3,\lambda} + b^{\alpha}_{\lambda}\bar{u}_{\alpha}) + [\bar{k}^{\beta\delta} + \\ & + \bar{h}^{2}({}'n^{\beta\delta} + {}''n^{\beta\delta})]\,(\bar{\psi}_{3,\delta} + b^{\alpha}_{\delta}\bar{\psi}_{\alpha}) + \bar{h}\,({}'m^{\beta\delta} - \\ & - \bar{h}\,{}'n^{\beta\delta})\,({}'\psi_{3,\delta} + b^{\alpha}_{\delta}\,{}''\psi_{\alpha}) - \bar{h}\,({}''m^{\beta\delta} + \bar{h}\,{}''n^{\beta\delta})\,({}''\psi_{3,\delta} + b^{\alpha}_{\beta}\,{}''\psi_{\alpha}) + \\ & + \bar{t}^{\beta}(1 + \bar{\psi}_{3}) + \bar{h}\,[{}'q^{\beta}(1 + {}'\psi_{3}) - {}''q^{\beta}(1 + {}''\psi_{3})] \Big\} \quad ,\end{aligned} \tag{138d}$$

$${}'\tilde{e}^{\lambda} = {}_{0}\nu_{\beta}\,[({}'m^{\beta\lambda} - \bar{h}\,{}'n^{\beta\lambda}) - ({}'k^{\beta\delta} - \bar{h}\,{}'m^{\beta\delta})\, b^{\lambda}_{\delta}] \qquad , \tag{138e}$$

$$\begin{aligned}{}'\tilde{e}^{3} = {}_{0}\nu_{\beta}\Big\{ & ({}'m^{\beta\delta} - \bar{h}\,{}'n^{\beta\delta})\,[\bar{u}_{3,\delta} + \bar{h}\,(\bar{\psi}_{3,\delta} - {}'\psi_{3,\delta}) + b^{\alpha}_{\delta}\bar{u}_{\alpha} + \\ & + \bar{h}\, b^{\alpha}_{\delta}(\bar{\psi}_{\alpha} - {}'\psi_{\alpha})] + ({}'k^{\beta\delta} - \bar{h}\,{}'m^{\beta\delta})\,({}'\psi_{3,\delta} + \\ & + b^{\alpha}_{\delta}\,{}'\psi_{\alpha}) + {}'t^{\beta}(1 + {}'\psi_{3}) - \bar{h}\,{}'q^{\beta}(1 + {}'\psi_{3}) \Big\} \quad ,\end{aligned} \tag{138f}$$

$${}''\tilde{e}^{\lambda} = {}_{0}\nu_{\beta}\,[({}''m^{\beta\lambda} + \bar{h}\,{}''n^{\beta\lambda}) - ({}''k^{\beta\delta} + \bar{h}\,{}''m^{\beta\delta})\, b^{\lambda}_{\delta}] \qquad , \tag{138g}$$

$$
{}^{*}\tilde{e}^{3} = {}_{0}\gamma_{\beta} \Big\{ ({}'m^{\beta\delta} + \bar{h}\,{}'n^{\beta\delta}) \big[\bar{u}_{3,\delta} - \bar{h}(\bar{\psi}_{3,\delta} - {}''\psi_{3,\delta}) + b^{\alpha}_{\delta}\bar{u}_{\alpha} - \bar{h}\, b^{\alpha}_{\delta}(\bar{\psi}_{\alpha} - {}'\psi_{\alpha}) \big] + ({}''k^{\beta\delta} + \bar{h}\,{}''m^{\beta\delta})({}''\psi_{3,\delta} + b^{\alpha}_{\delta}\,{}''\psi_{\alpha}) + {}''t^{\beta}(1 + {}''\psi_{3}) + \bar{h}\,{}''q^{\beta}(1 + {}''\psi_{3}) \Big\} \quad . \tag{138h}
$$

4.3 Analogy to Donnell-Mushtari-Vlasov Approximation

Simplification of the above equations is possible under the assumptions discussed, for example, by Mushtari and Galimov (19) and Novozhilov (20). This consists in neglecting the terms containing the product of $b^{\alpha}_{\beta} V^{*}_{\alpha}$ in the expression for $\gamma_{\alpha\beta}$ with the following results for the strains

$$
\gamma_{\alpha\beta} = \frac{1}{2}\big[\mu^{\delta}_{\alpha}(V^{*}_{\delta\|\beta} - b_{\delta\beta}V^{*}_{3}) + \mu^{\delta}_{\beta}(V^{*}_{\delta\|\alpha} - b_{\delta\alpha}V^{*}_{3}) + V^{*}_{3,\alpha}V^{*}_{3,\beta} \big] \quad , \tag{139a}
$$

$$
\gamma_{\alpha 3} = \frac{1}{2}\big[\mu^{\beta}_{\alpha} V^{*}_{\beta,3} + V^{*}_{3,\alpha} + b^{\delta}_{\alpha} V^{*}_{\delta} + V^{*}_{3,\alpha} V^{*}_{3,3} \big] \quad , \tag{139b}
$$

$$
\gamma_{33} = \frac{1}{2}\big[2V^{*}_{3,3} + V^{*}_{3,3}V^{*}_{3,3} \big] \quad . \tag{139c}
$$

The variational equation (86), compatible with the nonlinear strain-displacement relations (139) becomes

$$
\delta\Big[\int_{{}_{0}\tau} (s^{ij}\gamma_{ij} - \Phi)\,d\tau - \int_{{}_{0}\tau}\Big\{ \frac{1}{2} s^{\alpha\beta}\big[\mu^{\delta}_{\alpha}(V^{*}_{\delta\|\beta} - b_{\delta\beta}V^{*}_{3}) + \mu^{\delta}_{\beta}(V^{*}_{\delta\|\alpha} - b_{\delta\alpha}V^{*}_{3}) + V^{*}_{3,\alpha}V^{*}_{3,\beta}\big] + s^{\alpha 3}\big[\mu^{\beta}_{\alpha}V^{*}_{\beta,3} + V^{*}_{3,\alpha} + b^{\delta}_{\alpha}V^{*}_{\delta} + V^{*}_{3,\alpha}V^{*}_{3,3}\big] + \frac{1}{2}s^{33}\big[2V^{*}_{3,3} + V^{*}_{3,3}V^{*}_{3,3}\big]\Big\}\,d\tau + \int_{{}_{0}\tau}\big[\rho_{0}({}_{0}b^{\alpha} - {}_{0}a^{\alpha})\mu^{\delta}_{\alpha}V^{*}_{\delta} + \rho_{0}({}_{0}b^{3} - {}_{0}a^{3})V^{*}_{3}\big]d\tau + \int_{{}_{0}S}\big[({}_{0}n_{\beta}\tilde{t}^{\beta\alpha} + {}_{0}n_{3}\tilde{t}^{3\alpha})\mu^{\delta}_{\alpha}V^{*}_{\delta} + ({}_{0}n_{\beta}\tilde{t}^{\beta 3} + {}_{0}n_{3}\tilde{t}^{33}) \times V^{*}_{3}\big]\,dS \Big] = 0 \quad , \tag{140}
$$

$$\delta I_1 + \delta I_2 + \delta I_3 + \delta J = 0 \quad , \tag{141}$$

where

$$\delta I_1 = \int_{0\tau} \Big[\{ (\mu \mu^\delta_\alpha s^{\alpha\beta})_{\|\beta} - \mu s^{\beta 3} b^\delta_\beta + (\mu \mu^\delta_\alpha s^{\alpha 3})_{,3} + \mu \mu^\delta_\alpha \rho_0 ({}_0b^\alpha - {}_0a^\alpha) \} \delta V^*_\delta + \{ \mu \mu^\delta_\alpha s^{\alpha\beta} b_{\delta\beta} + [\mu s^{\alpha\beta} V^*_{3,\alpha} + \mu s^{\beta 3}(1 + V^*_{3,3})]_{\|\beta} + [\mu s^{\alpha 3} V^*_{3,\alpha} + \mu s^{33}(1+V^*_{3,3})]_{,3} + \mu \rho_0 ({}_0b^3 - {}_0a^3) \} \delta V^*_3 \Big] d\theta^3 \, dA \quad , \tag{142a}$$

$$\delta I_2 = \iint_{0\tau} \Big[\{ \gamma_{\alpha\beta} - \tfrac{1}{2}[\mu^\delta_\alpha (V^*_{\delta\|\beta} - b_{\delta\beta} V^*_3) + \mu^\delta_\beta (V^*_{\delta\|\alpha} - b_{\delta\alpha} V^*_3) + V^*_{3,\alpha} V^*_{3,\beta}] \} \delta s^{\alpha\beta} + \{ 2\gamma_{\alpha 3} - [\mu^\beta_\alpha V^*_{\beta,3} + V^*_{3,\alpha} + b^\delta_\alpha V^*_\delta + V^*_{3,\alpha} V^*_{3,3}] \} \delta s^{\alpha 3} + \{ \gamma_{33} - \tfrac{1}{2}[2 V^*_{3,3} + V^*_{3,3} V^*_{3,3}] \} \delta s^{33} \Big] \mu \, dA \, d\theta^3 \quad , \tag{142b}$$

$$\delta I_3 = \int_{0\tau} [s^{ij} - \tfrac{1}{2} (\frac{\partial \Phi}{\partial \gamma_{ij}} - \frac{\partial \Phi}{\partial \gamma_{ji}})] \, \delta \gamma_{ij} \, \mu \, d\theta^3 dA \quad , \tag{142c}$$

$$\delta J = \int_{0C} \int_{-h}^{h} \{ {}_0\nu_\beta (\mu \mu^\delta_\alpha \tilde{t}^{\beta\alpha} - \mu \mu^\delta_\alpha s^{\alpha\beta}) \delta V^*_\delta + {}_0\nu_\beta [\mu \tilde{t}^{\beta 3} - \mu s^{\alpha\beta} V^*_{3,\alpha} - \mu s^{\alpha 3}(1 + V^*_{3,3})] \delta V^*_3 \} d\theta^3 ds + \int_{0A} \{ {}_0n_3 (\mu \mu^\delta_\alpha \tilde{t}^{3\alpha} - \mu \mu^\delta_\alpha s^{\alpha 3}) \delta V^*_\delta + {}_0n_3 [\mu \tilde{t}^{33} - \mu s^{\alpha 3} V^*_{3,\alpha} - \mu s^{33}(1 + V^*_{3,3})] \delta V^*_3 \} dA \quad . \tag{142d}$$

By the procedure used in Chapter III, we obtain the corresponding nonlinear equations of motion as follows.

$$\delta \bar{u}_\delta : \quad (n^{\alpha\delta} - m^{\alpha\beta} b^\delta_\beta)_{\|\alpha} - q^\alpha b^\delta_\alpha + p^\delta - f^\delta = 0 \quad , \tag{143a}$$

$$\delta \bar{u}_3 : \quad \{ n^{\alpha\beta} \bar{u}_{3,\beta} + [\bar{m}^{\alpha\beta} + \bar{h}('n^{\alpha\beta} - ''n^{\alpha\beta})] \bar{\psi}_{3,\beta} + ('m^{\alpha\beta} - \bar{h}'n^{\alpha\beta})'\psi_{3,\beta} +$$
$$+ (''m^{\alpha\beta} + \bar{h}''n^{\alpha\beta}) ''\psi_{3,\beta} + q^\alpha + 'q^\alpha \, '\psi_3 + \bar{q}^\alpha \bar{\psi}_3 + ''q^\alpha \, ''\psi_3 \}\|_\alpha +$$
$$+ (n^{\alpha\beta} - m^{\alpha\delta} b^\beta_\delta) b_{\alpha\beta} + p^3 - f^3 = 0 \quad , \tag{143b}$$

$$\delta \bar{\psi}_\delta : \quad \{ [\bar{m}^{\alpha\delta} + \bar{h}('n^{\alpha\delta} - ''n^{\alpha\delta})] - [\bar{k}^{\alpha\beta} + \bar{h}('n^{\alpha\beta} - ''n^{\alpha\beta})] \, b^\delta_\beta \}\|_\alpha +$$
$$+ ('q^\alpha - ''q^\alpha) \bar{h} \, b^\delta_\alpha - \bar{q}^\delta + c^\delta - m^\delta = 0 \quad , \tag{143c}$$

$$\delta \bar{\psi}_3 : \quad \{ [\bar{m}^{\alpha\beta} + \bar{h}('n^{\alpha\beta} - ''n^{\alpha\beta})] \, \bar{u}_{3,\beta} + [\bar{k}^{\alpha\beta} + \bar{h}^2('n^{\alpha\beta} + ''n^{\alpha\beta}) \bar{\psi}_{3,\beta} +$$
$$+ \bar{h}('m^{\alpha\beta} - \bar{h}'n^{\alpha\beta}) \, '\psi_{3,\beta} - \bar{h}(''m^{\alpha\beta} + \bar{h}''n^{\alpha\beta}) \, ''\psi_{3,\beta} +$$
$$+ ('q^\alpha - ''q^\alpha) \bar{h} \}\|_\alpha + \{ [\bar{m}^{\beta\delta} + \bar{h}('n^{\beta\delta} - ''n^{\beta\delta})] - [\bar{k}^{\beta\alpha} +$$
$$+ \bar{h}('m^{\beta\alpha} - ''m^{\beta\alpha})] \, b^\delta_\alpha \} \, b_{\delta\beta} - \bar{q}^\alpha \bar{u}_{3,\alpha} + (\bar{t}^\alpha\|_\alpha -$$
$$- \bar{n}^{33}) (1 + \bar{\psi}_3) + c^3 - m^3 = 0 \quad , \tag{143d}$$

$$\delta '\psi_\delta : \quad [('m^{\alpha\delta} - \bar{h}'n^{\alpha\delta}) - ('k^{\alpha\beta} - \bar{h}'m^{\alpha\beta}) \, b^\delta_\beta]\|_\alpha + 'q^\alpha b^\delta_\alpha \bar{h} -$$
$$- 'q^\delta + 'g^\delta - 'd^\delta = 0 \quad , \tag{143e}$$

$$\delta '\psi_3 : \quad \{ ('m^{\alpha\beta} - \bar{h}'n^{\alpha\beta}) [\bar{u}_{3,\beta} + \bar{h}(\bar{\psi}_{3,\beta} - '\psi_{3,\beta})] + ('k^{\alpha\beta} -$$
$$- \bar{h}'m^{\alpha\beta}) \, '\psi_{3,\beta} - 'q^\alpha \bar{h} (1 + '\psi_3) \}\|_\alpha + [('m^{\beta\alpha} - \bar{h}'n^{\beta\alpha}) -$$
$$- ('k^{\beta\delta} - \bar{h}'m^{\beta\delta}) \, b^\alpha_\delta] \, b_{\beta\alpha} - 'q^\alpha [\bar{u}_{3,\alpha} + \bar{h}(\bar{\psi}_{3,\alpha} - '\psi_{3,\alpha})] +$$
$$+ ('t^\alpha\|_\alpha - 'n^{33}) (1 + '\psi_3) + 'g^3 - 'd^3 = 0 \quad , \tag{143f}$$

$$\delta ''\psi_\delta : \quad [(''m^{\alpha\delta} + \bar{h}''n^{\alpha\delta}) + (''k^{\alpha\beta} + \bar{h}''m^{\alpha\beta}) \, b^\delta_\beta]\|_\alpha - ''q^\alpha \bar{h} \, b^\delta_\alpha -$$
$$- ''q^\delta + ''g^\delta + ''d^\delta = 0 \quad , \tag{143g}$$

$$\delta ''\psi_3 : \quad \{ (''m^{\alpha\beta} + \bar{h}''n^{\alpha\beta}) [\bar{u}_{3,\beta} - \bar{h}(\bar{\psi}_{3,\beta} - ''\psi_{3,\beta})] + (''k^{\alpha\beta} +$$
$$+ \bar{h}''m^{\alpha\beta}) \, ''\psi_{3,\beta} + ''q^\alpha \bar{h} (1 + ''\psi_3) \}\|_\alpha + (''m^{\beta\alpha} +$$

$$+ \bar{h}\,''n^{\beta\alpha}) - ({}^{*}k^{\beta\delta} + \bar{h}\,'m^{\beta\delta})\, b^{\alpha}_{\delta} \,]\, b_{\beta\alpha} - \,''q^{\alpha}[\,\bar{u}_{3,\alpha} -$$

$$- \bar{h}\,(\bar{\psi}_{3,\alpha} - {}''\psi_{3,\alpha})] + ({}''t^{\alpha}\|_{\alpha} - {}''n^{33})\,(1 + {}''\psi_3) +$$

$$+ {}''g^3 + {}^{*}d^3 = 0 \quad . \tag{143h}$$

The corresponding boundary conditions are to prescribe

$$\tilde{p}^i_+ = {}'p^i_+ \qquad , \qquad -\tilde{p}^i_- = {}''p^i_- \quad ,$$

and

$$\tilde{S}^{\lambda} = {}_{\circ}\nu_{\beta}\,(n^{\beta\lambda} - m^{\beta\delta} b^{\lambda}_{\delta}) \qquad , \tag{144a}$$

$$\tilde{S}^{3} = {}_{\circ}\nu_{\beta}\,\{\, n^{\beta\delta}\bar{u}_{3,\delta} + [\bar{m}^{\beta\delta} + \bar{h}\,({}'n^{\beta\delta} - {}''n^{\beta\delta})]\,\bar{\psi}_{3,\delta} +$$

$$+ ({}'m^{\beta\delta} - \bar{h}\,'n^{\beta\delta})\,{}'\psi_{3,\delta} + ({}''m^{\beta\delta} + \bar{h}\,''n^{\beta\delta})\,{}''\psi_{3,\delta} +$$

$$+ q^{\beta} + {}'q^{\beta}\,{}'\psi_3 + \bar{q}^{\beta}\bar{\psi}_3 + {}''q^{\beta}\,{}''\psi_3 \,\} \qquad , \tag{144b}$$

$$\tilde{l}^{\lambda} = {}_{\circ}\nu_{\beta}\,\{[\bar{m}^{\beta\lambda} + \bar{h}\,({}'n^{\beta\lambda} - {}''n^{\beta\lambda})] - [\,\bar{k}^{\alpha\beta} + \bar{h}\,({}'m^{\alpha\beta} - {}''m^{\alpha\beta})]\, b^{\lambda}_{\alpha}\,\} \qquad , \tag{144c}$$

$$\tilde{l}^{3} = {}_{\circ}\nu_{\beta}\,\{[\,\bar{m}^{\beta\delta} + \bar{h}\,({}'n^{\beta\delta} - {}''n^{\beta\delta})]\,\bar{u}_{3,\delta} + [\,\bar{k}^{\beta\delta} + \bar{h}^2({}'n^{\beta\delta} +$$

$$+ {}''n^{\beta\delta})]\,\bar{\psi}_{3,\delta} + \bar{h}\,({}'m^{\beta\delta} - \bar{h}\,'n^{\beta\delta})\,{}'\psi_{3,\delta} - \bar{h}\,({}''m^{\beta\delta} +$$

$$+ \bar{h}\,''n^{\beta\delta})\,{}''\psi_{3,\delta} + \bar{t}^{\beta}(1 + \bar{\psi}_3) + \bar{h}\,({}'q^{\beta} - {}''q^{\beta})\} \quad , \tag{144d}$$

$${}'\tilde{e}^{\lambda} = {}_{\circ}\nu_{\beta}\,[\,({}'m^{\beta\lambda} - \bar{h}\,'n^{\beta\lambda}) - ({}'k^{\beta\delta} - \bar{h}\,'m^{\beta\delta})\, b^{\lambda}_{\delta}\,] \qquad , \tag{144e}$$

$${}'\tilde{e}^{3} = {}_{\circ}\nu_{\beta}\,\{\,({}'m^{\beta\delta} - \bar{h}\,'n^{\beta\delta})\,[\,\bar{u}_{3,\delta} + \bar{h}\,(\bar{\psi}_{3,\delta} - {}'\psi_{3,\delta})] + ({}'k^{\beta\delta} -$$

$$- \bar{h}\,'m^{\beta\delta})\,{}'\psi_{3,\delta} + {}'t^{\beta}(1 + {}'\psi_3) - \bar{h}\,'q^{\beta}\,\} \quad , \tag{144f}$$

$${}^{*}\tilde{e}^{\lambda} = {}_{\circ}\nu_{\beta}\,[\,({}^{*}m^{\beta\lambda} + \bar{h}\,{}^{*}n^{\beta\lambda}) + ({}^{*}k^{\beta\delta} + \bar{h}\,{}^{*}n^{\beta\delta})\, b^{\lambda}_{\delta}\,] \quad , \tag{144g}$$

$$''\tilde{e}^{3} = {}_{\bullet}\gamma_{\beta} \Big\{ (''m^{\beta\delta} + \bar{h}\,''n^{\beta\delta}) \left[\bar{u}_{3,\delta} - \bar{h}(\bar{\psi}_{3,\delta} - ''\psi_{3,\delta}) \right] + (''k^{\beta\delta} + \bar{h}\,''m^{\beta\delta})\,''\psi_{3,\delta} + ''t^{\beta}(1 + ''\psi_{3}) + \bar{h}\,''q^{\beta} \Big\} , \tag{144h}$$

where

$$\begin{Bmatrix} 'p^{3}_{\pm} \\ 'C^{3}_{\pm} \end{Bmatrix} = \left[\left\{ \mu\, 'S^{3\beta}(\bar{u}_{3,\beta} + \bar{h}\bar{\psi}_{3,\beta} - \bar{h}\,'\psi_{3,\beta}) + \mu\, 'S^{33}(1 + '\psi_{3}) \right\} \begin{Bmatrix} 1 \\ \theta^{3} \end{Bmatrix} \right]_{\theta^{3} = 'z \pm 'h} ,$$

and $'p^{\alpha}_{\pm}$, $'C^{\alpha}_{\pm}$ are defined in Equations (136).

4.4 Partially Nonlinear Membrane Theory of Sandwich Shells

When the facings are so thin that they are effectively membranes, we set

$$'\psi_{\alpha} = '\psi_{3} = ''\psi_{\alpha} = ''\psi_{3} = 0 ,$$

and

$$'S^{\alpha 3} = ''S^{\alpha 3} = 0 ,$$

in Equations (143) and (144). Accordingly, we may suppress the equations corresponding to the variations $\delta\,'\psi_{\delta}$, $\delta\,'\psi_{3}$, $\delta\,''\psi_{\delta}$, and $\delta\,''\psi_{3}$. The resulting equations can be simplified considerably for sandwich shells. The equations of motion are

$$\delta\bar{u}_{\delta} : \quad (n^{\alpha\delta} - m^{\alpha\beta} b^{\delta}_{\beta})_{\|\alpha} - \bar{q}^{\alpha} b^{\delta}_{\alpha} + p^{\delta} - f^{\delta} = 0 , \tag{145a}$$

$$\begin{aligned} \delta\bar{u}_{3} : \quad & \big\{ n^{\alpha\beta}\bar{u}_{3,\beta} + [\bar{m}^{\alpha\beta} + \bar{h}('n^{\alpha\beta} - ''n^{\alpha\beta})]\bar{\psi}_{3,\beta} + \\ & + \bar{q}^{\alpha}(1 + \bar{\psi}_{3}) \big\}_{\|\alpha} + (n^{\alpha\beta} + m^{\alpha\delta} b^{\beta}_{\delta}) b_{\alpha\beta} + \\ & + p^{3} - f^{3} = 0 , \end{aligned} \tag{145b}$$

$$\delta\bar{\psi}_\delta : \quad \left\{ \left[\bar{m}^{\alpha\delta} + \bar{h}\left({}'n^{\alpha\delta} - {}''n^{\alpha\delta}\right) \right] - \left[\bar{k}^{\alpha\beta} + \bar{h}\left({}'m^{\alpha\beta} - {}''m^{\alpha\beta}\right) b_\beta^\delta \right\}_{\|\alpha} -$$

$$- \bar{q}^\delta + c^\delta - m^\delta = 0 \quad , \tag{145c}$$

$$\delta\bar{\psi}_3 : \quad \left\{ \left[\bar{m}^{\alpha\beta} + \bar{h}\left({}'n^{\alpha\beta} - {}''n^{\alpha\beta}\right) \right] \bar{u}_{3,\beta} + \left[\bar{k}^{\alpha\beta} + \bar{h}^2\left({}'n^{\alpha\beta} + {}''n^{\alpha\beta}\right) \right] \bar{\psi}_{3,\beta} \right\}_{\|\alpha} +$$

$$+ \left\{ \left[\bar{m}^{\beta\delta} + \bar{h}\left({}'n^{\beta\delta} - {}''n^{\beta\delta}\right) \right] - \left[\bar{k}^{\beta\alpha} + \bar{h}\left({}'m^{\alpha\beta} - {}''m^{\alpha\beta}\right) \right] b_\alpha^\delta \right\} b_{\delta\beta} -$$

$$- \bar{q}^\alpha \bar{u}_{3,\alpha} + \left(\bar{t}^\alpha{}_{\|\alpha} - \bar{n}^{33} \right)\left(1 + \bar{\psi}_3\right) + c^3 - m^3 = 0 \quad . \tag{145d}$$

The edge boundary conditions are

$$\tilde{S}^\lambda = {}_0\nu_\beta \left(n^{\beta\lambda} - m^{\beta\delta} b_\delta^\lambda \right) \quad , \tag{146a}$$

$$\tilde{S}^3 = {}_0\nu_\beta \left\{ n^{\beta\delta} \bar{u}_{3,\delta} + \left[\bar{m}^{\beta\delta} + \bar{h}\left({}'n^{\beta\delta} - {}''n^{\beta\delta}\right) \right] \bar{\psi}_{3,\delta} + \right.$$

$$\left. + \bar{q}^\beta \left(1 + \bar{\psi}_3\right) \right\} \quad , \tag{146b}$$

$$\tilde{\ell}^\lambda = {}_0\nu_\beta \left\{ \left[\bar{m}^{\beta\lambda} + \bar{h}\left({}'n^{\beta\lambda} - {}''n^{\beta\lambda}\right) \right] - \left[\bar{k}^{\beta\alpha} + \bar{h}\left({}'m^{\beta\alpha} - {}''m^{\beta\alpha}\right) \right] b_\alpha^\lambda \right\} \quad , \tag{146c}$$

$$\tilde{\ell}^3 = {}_0\nu_\beta \left\{ \left[\bar{m}^{\beta\delta} + \bar{h}\left({}'n^{\beta\delta} - {}''n^{\beta\delta}\right) \right] \bar{u}_{3,\delta} + \left[\bar{k}^{\beta\delta} + \bar{h}^2\left({}'n^{\beta\delta} + \right.\right.\right.$$

$$\left.\left.\left. + {}''n^{\beta\delta}\right) \right] \bar{\psi}_{3,\delta} + \bar{t}^\beta \left(1 + \bar{\psi}_3\right) \right\} \quad . \tag{146d}$$

4.5 Sandwich Shells with a Weak Core

Assuming that the components of stress $\bar{S}^{\alpha\beta}$ in the core are of negligible importance, they may be set equal to zero, and the components of transverse shear and normal stress, $\bar{S}^{\alpha 3}$ and $\bar{S}^{33}$, only are retained. The equations of motion (145) and boundary conditions (146) then become

$$\delta\bar{u}_\delta : \quad \left(n^{\alpha\delta} - m^{\alpha\beta} b_\beta^\delta \right)_{\|\alpha} - \bar{q}^\alpha b_\alpha^\delta + p^\delta - f^\delta = 0 \quad , \tag{147a}$$

$$\delta \bar{u}_3 : \quad \left[n^{\alpha\beta} \bar{u}_{3,\beta} + \bar{h}\,({}'n^{\alpha\beta} - {}''n^{\alpha\beta})\, \bar{\Psi}_{3,\beta} + \bar{q}^{\alpha}(1+\bar{\Psi}_3) \right]_{\|\alpha} +$$

$$+ (n^{\alpha\beta} + m^{\alpha\delta} b_{\delta}^{\beta})\, b_{\alpha\beta} + p^3 - f^3 = 0 \quad , \qquad (147b)$$

$$\delta \bar{\Psi}_{\delta} : \quad \bar{h}\left[({}'n^{\alpha\delta} - {}''n^{\alpha\delta}) - ({}'m^{\alpha\beta} - {}''m^{\alpha\beta})\, b_{\beta}^{\delta} \right]_{\|\alpha} - \bar{q}^{\delta} + c^{\delta} - m^{\delta} = 0 \quad , \qquad (147c)$$

$$\delta \bar{\Psi}_3 : \quad \left[\bar{h}\,({}'n^{\alpha\beta} - {}''n^{\alpha\beta})\, \bar{u}_{3,\beta} + \bar{h}^2 ({}'n^{\alpha\beta} + {}''n^{\alpha\beta})\, \bar{\Psi}_{3,\beta} \right]_{\|\alpha} +$$

$$+ \bar{h}\left[({}'n^{\alpha\beta} - {}''n^{\alpha\beta}) - ({}'m^{\alpha\delta} - {}''m^{\alpha\delta})\, b_{\delta}^{\beta} \right] b_{\alpha\beta} - \bar{q}^{\alpha} \bar{u}_{3,\alpha} +$$

$$+ (\bar{t}^{\alpha}\|_{\alpha} - \bar{n}^{33})(1+\bar{\Psi}_3) + c^3 - m^3 = 0 \quad , \qquad (147d)$$

and

$$\tilde{S}^{\lambda} = {}_0\nu_{\beta}\, (n^{\beta\lambda} - m^{\beta\delta} b_{\delta}^{\lambda}) \quad , \qquad (148a)$$

$$\tilde{S}^{3} = {}_0\nu_{\beta} \left\{ n^{\beta\delta} \bar{u}_{3,\delta} + [\bar{m}^{\beta\delta} + \bar{h}\,({}'n^{\beta\delta} - {}''n^{\beta\delta})]\, \bar{\Psi}_{3,\delta} + \right.$$

$$\left. + \bar{q}^{\beta}(1+\bar{\Psi}_3) \right\} \quad , \qquad (148b)$$

$$\tilde{L}^{\lambda} = {}_0\nu_{\beta} \left\{ \bar{h}\left[({}'n^{\beta\lambda} - {}''n^{\beta\lambda}) - ({}'m^{\beta\delta} - {}''m^{\beta\delta})\, b_{\delta}^{\lambda} \right] \right\} \quad , \qquad (148c)$$

$$\tilde{L}^{3} = {}_0\nu_{\beta} \left\{ \bar{h}\,({}'n^{\beta\delta} - {}''n^{\beta\delta})\, \bar{u}_{3,\delta} + \bar{h}^2 ({}'n^{\beta\delta} - {}''n^{\beta\delta})\, \bar{\Psi}_{3,\delta} + \right.$$

$$\left. + \bar{t}^{\beta}(1+\bar{\Psi}_3) \right\} \quad , \qquad (148d)$$

respectively, where the definitions of $n^{\alpha\beta}$, $m^{\alpha\beta}$, p^i , and f^i are also simplified accordingly.

4.6 Linearization of the General Equations of Motion

Dropping all nonlinear terms in the equations of motion

(105) and in the boundary conditions (122), we obtain the following equations of motion

$$\delta \bar{u}_\delta : \quad (n^{\alpha\delta} - m^{\alpha\rho} b^\delta_\rho)\|_\alpha - q^\alpha b^\delta_\alpha + p^\delta - f^\delta = 0 \quad , \tag{149a}$$

$$\delta \bar{u}_3 : \quad q^\alpha\|_\alpha + (n^{\beta\alpha} - m^{\beta\delta} b^\alpha_\delta) b_{\alpha\beta} + p^3 - f^3 = 0 \quad , \tag{149b}$$

$$\delta \bar{\psi}_\delta : \quad \left\{ [\bar{m}^{\alpha\delta} + \bar{h}({}'n^{\alpha\delta} - {}''n^{\alpha\delta})] - [\bar{k}^{\alpha\rho} + \bar{h}({}'m^{\alpha\rho} - {}''m^{\alpha\rho})] b^\delta_\rho \right\}\|_\alpha -$$

$$- ({}'q^\alpha - {}''q^\alpha)\bar{h} b^\delta_\alpha - \bar{q}^\delta + c^\delta - m^\delta = 0 \quad , \tag{149c}$$

$$\delta \bar{\psi}_3 : \quad [\bar{t}^\alpha + \bar{h}({}'q^\alpha - {}''q^\alpha)]\|_\alpha + \left\{ [\bar{m}^{\alpha\delta} + \bar{h}({}'n^{\alpha\delta} - {}''n^{\alpha\delta})] - \right.$$

$$\left. - [\bar{k}^{\alpha\rho} + \bar{h}({}'m^{\alpha\rho} - {}''m^{\alpha\rho})] b^\delta_\rho \right\} b_{\delta\alpha} - \bar{n}^{33} +$$

$$+ c^3 - m^3 = 0 \quad , \tag{149d}$$

$$\delta {}'\psi_\delta : \quad [({}'m^{\alpha\delta} - \bar{h}\,{}'n^{\alpha\delta}) - ({}'k^{\alpha\rho} - \bar{h}\,{}'m^{\alpha\rho}) b^\delta_\rho]\|_\alpha + {}'q^\alpha \bar{h} b^\delta_\alpha -$$

$$- {}'q^\delta + {}'g^\delta - {}'d^\delta = 0 \quad , \tag{149e}$$

$$\delta {}'\psi_3 : \quad ({}'t^\alpha - {}'q^\alpha \bar{h})\|_\alpha - {}'n^{33} + {}'g^3 - {}'d^3 = 0 \quad , \tag{149f}$$

$$\delta {}''\psi_\delta : \quad [({}''m^{\alpha\delta} + \bar{h}\,{}''n^{\alpha\delta}) - ({}''k^{\alpha\rho} + \bar{h}\,{}''m^{\alpha\rho}) b^\delta_\rho]\|_\alpha - {}''q^\alpha \bar{h} -$$

$$- {}''q^\delta + {}''g^\delta + {}''d^\delta = 0 \quad , \tag{149g}$$

$$\delta {}''\psi_3 : \quad ({}''t^\alpha + {}''q^\alpha \bar{h})\|_\alpha - {}''n^{33} + {}''g^3 + {}''d^3 = 0 \quad , \tag{149h}$$

and boundary conditions

$$\tilde{S}^\lambda = {}_\circ\nu_\beta (n^{\beta\lambda} - m^{\beta\delta} b^\lambda_\delta) \quad , \tag{150a}$$

$$\tilde{S}^3 = {}_0\nu_\beta \, q^\beta \quad , \tag{150b}$$

$$\tilde{l}^\lambda = {}_0\nu_\beta \left\{ \left[\bar{m}^{\beta\lambda} + \bar{h}\left({}'n^{\beta\lambda} - {}''n^{\beta\lambda}\right) \right] - \left[\bar{k}^{\beta\delta} + \bar{h}\left({}'m^{\beta\delta} - {}''m^{\beta\delta}\right) \right] b_\delta^\lambda \right\} , \tag{150c}$$

$$\tilde{l}^3 = {}_0\nu_\beta \left[\bar{t}^\alpha + \bar{h}\left({}'q^\alpha - {}''q^\alpha\right) \right] \quad , \tag{150d}$$

$${}'\tilde{e}^\lambda = {}_0\nu_\beta \left[\left({}'m^{\beta\lambda} - \bar{h}\,{}'n^{\beta\lambda}\right) - \left({}'k^{\beta\delta} - \bar{h}\,{}'m^{\beta\delta}\right) b_\delta^\lambda \right] \quad , \tag{150e}$$

$${}'\tilde{e}^3 = {}_0\nu_\beta \left({}'t^\beta - \bar{h}\,{}'q^\beta\right) \quad , \tag{150f}$$

$${}''\tilde{e}^\lambda = {}_0\nu_\beta \left[\left({}''m^{\beta\lambda} + \bar{h}\,{}''n^{\beta\lambda}\right) - \left({}''k^{\beta\delta} - \bar{h}\,{}''m^{\beta\delta}\right) b_\delta^\lambda \right] \quad , \tag{150g}$$

$${}''\tilde{e}^3 = {}_0\nu_\beta \left({}''t^\beta + \bar{h}\,{}''q^\beta\right) \quad . \tag{150h}$$

CHAPTER V

COMPARISONS AND CONCLUSION

The nonlinear fundamental equations of a theory of sandwich shells in terms of the undeformed state have been obtained from three-dimensional continuum mechanics with the help of the modified Hellinger-Reissner variational theorem. These include strain-displacement relations, equations of motion, boundary conditions, and constitutive equations for elastic, anisotropic, composite shells subjected to large displacement gradients under the influence of mechanical and thermal loads. Several approximate systems of equations which may be suitable for application to cases in which the displacements and rotations are restricted in magnitude have also been obtained.

For a sandwich plate, μ^{α}_{β} is simply the Kronecker symbol, since then $b_{\alpha\beta} = 0$. Equations (105) become

$$\delta \bar{u}_{\delta} : \quad \Big\{ n^{\alpha\beta} (\delta^{\delta}_{\beta} + \bar{u}^{\delta}|_{\beta}) + [\bar{m}^{\alpha\beta} + \bar{h} ('n^{\alpha\beta} - ''n^{\alpha\beta})] \bar{\psi}^{\delta}|_{\beta} +$$

$$+ ('m^{\alpha\beta} - \bar{h}\, 'n^{\alpha\beta})\, '\psi^{\delta}|_{\beta} + (''m^{\alpha\beta} + \bar{h}\, ''n^{\alpha\beta})\, ''\psi^{\delta}|_{\beta} +$$

$$+ 'q^{\alpha}\, '\psi^{\delta} + \bar{q}^{\alpha} \bar{\psi}^{\delta} + ''q^{\alpha}\, ''\psi^{\delta} \Big\}\Big|_{\alpha} + p^{\delta} - f^{\delta} = 0 \,, \tag{151a}$$

$$\delta \bar{u}_{3} : \quad \Big\{ n^{\alpha\beta} \bar{u}_{3,\beta} + [\bar{m}^{\alpha\beta} + \bar{h} ('n^{\alpha\beta} - ''n^{\alpha\beta})] \bar{\psi}_{3,\beta} +$$

$$+ ('m^{\alpha\beta} - \bar{h}\, 'n^{\alpha\beta})\, '\psi_{3,\beta} + (''m^{\alpha\beta} + \bar{h}\, ''n^{\alpha\beta})\, ''\psi_{3,\beta} + q^{\alpha} +$$

$$+ 'q^{\alpha}\, '\psi_{3} + \bar{q}^{\alpha} \bar{\psi}_{3} + ''q^{\alpha}\, ''\psi_{3} \Big\}\Big|_{\alpha} + p^{3} - f^{3} = 0 \,, \tag{151b}$$

$$\begin{aligned}\delta\bar\psi_\delta :\quad & \Big\{\big[\bar m^{\alpha\beta} + \bar h\,({}'n^{\alpha\beta} - {}''n^{\alpha\beta})\big](\delta^\delta_\beta + \bar u^\delta|_\beta) - \big[\bar k^{\alpha\beta} + \bar h^2\,({}'n^{\alpha\beta} +\\ & + {}''n^{\alpha\beta})\big]\bar\psi^\delta|_\beta + \bar h\,({}'m^{\alpha\beta} - \bar h\,{}'n^{\alpha\beta})\,{}'\psi^\delta|_\beta - \bar h\,({}''m^{\alpha\beta} +\\ & + \bar h\,{}''n^{\alpha\beta})\,{}''\psi^\delta|_\beta + \bar h\,({}'q^\alpha\,{}'\psi^\delta - {}''q^\alpha\,{}''\psi^\delta)\Big\}\Big|_\alpha + (\bar t^\alpha|_\alpha -\\ & - \bar n^{33}) - \bar q^\alpha(\delta^\delta_\alpha + \bar u^\delta|_\alpha) + c^\delta - m^\delta = 0 \quad ,\end{aligned} \tag{151c}$$

$$\begin{aligned}\delta\bar\psi_3 :\quad & \Big\{\big[\bar m^{\alpha\beta} + \bar h\,({}'n^{\alpha\beta} - {}''n^{\alpha\beta})\big]\,\bar u_{3,\beta} + \big[\bar k^{\alpha\beta} +\\ & + \bar h^2\,({}'n^{\alpha\beta} + {}''n^{\alpha\beta})\big]\,\bar\psi_{3,\beta} + \bar h\,({}'m^{\alpha\beta} - \bar h\,{}'n^{\alpha\beta})\,{}'\psi_{3,\beta} -\\ & - \bar h\,({}''m^{\alpha\beta} + \bar h\,{}''n^{\alpha\beta})\,{}''\psi_{3,\beta} + \big[\,{}'q^\alpha(1 + {}'\psi_3) -\\ & - {}''q^\alpha(1 + {}''\psi_3)\,\bar h\,\Big\}\Big|_\alpha - \bar q^\alpha\,\bar u_{3,\alpha} + (\bar t^\alpha|_\alpha -\\ & - \bar n^{33}(1 + \bar\psi_3) + c^3 - m^3 = 0 \quad ,\end{aligned} \tag{151d}$$

$$\begin{aligned}\delta\,{}'\psi_\delta :\quad & \Big\{({}'m^{\alpha\beta} - \bar h\,{}'n^{\alpha\beta})\big[\delta^\delta_\beta + \bar u^\delta|_\beta + \bar h\,(\bar\psi^\delta|_\beta - {}'\psi^\delta|_\beta)\big] +\\ & + ({}'k^{\alpha\beta} - \bar h\,{}'m^{\alpha\beta})\,{}'\psi^\delta|_\beta - \bar h\,{}'q^\alpha\,{}'\psi^\delta\Big\}\Big|_\alpha +\\ & + (\bar t^\alpha|_\alpha - \bar n^{33})\,{}'\psi^\delta - {}'q^\delta - {}'q^\alpha\big[\bar u^\delta_{,\alpha} + \bar h\,(\bar\psi^\delta|_\alpha - {}'\psi^\delta|_\alpha)\big] +\\ & + {}'g^\delta - {}'d^\delta = 0 \quad ,\end{aligned} \tag{151e}$$

$$\begin{aligned}\delta\,{}'\psi_3 :\quad & \Big\{({}'m^{\alpha\beta} - \bar h\,{}'n^{\alpha\beta})\big[\bar u_{3,\beta} + \bar h\,(\bar\psi_{3,\beta} - {}'\psi_{3,\beta})\big] +\\ & + ({}'k^{\alpha\beta} - \bar h\,{}'m^{\alpha\beta})\,{}'\psi_{3,\beta} - {}'q^\alpha\,\bar h\,(1 + {}'\psi_3)\Big\}\Big|_\alpha +\\ & + ({}'t^\alpha|_\alpha - {}'n^{33})(1 + {}'\psi_3) - {}'q^\alpha\big[\bar u_{3,\alpha} + \bar h\,(\bar\psi_{3,\alpha} -\\ & - {}'\psi_{3,\alpha})\big] + {}'g^3 - {}'d^3 = 0 \quad ,\end{aligned} \tag{151f}$$

$$\begin{aligned}
\delta\,''\psi_\delta : \quad & \Big\{ (''m^{\alpha\beta} + \bar{h}\,''n^{\alpha\beta}) \left[\delta^\delta_\beta + \bar{u}^\delta|_\beta - \bar{h}(\bar{\psi}^\delta|_\beta - ''\psi^\delta|_\beta) \right] + \\
& + (''k^{\alpha\beta} + \bar{h}\,''m^{\alpha\beta})\,''\psi^\delta|_\beta + \bar{h}\,''q^\alpha\,''\psi^\delta \Big\}\Big|_\alpha + \\
& + (''t^\alpha|_\alpha - ''n^{33})\,''\psi^\delta - ''q^\alpha \left[\delta^\delta_\alpha + \bar{u}^\delta|_\alpha - \bar{h}(\bar{\psi}^\delta|_\alpha - ''\psi^\delta|_\alpha) \right] + \\
& + ''g^\delta - ''d^\delta = 0 \quad ,
\end{aligned} \tag{151g}$$

$$\begin{aligned}
\delta\,''\psi_3 : \quad & \Big\{ (''m^{\alpha\beta} + \bar{h}\,''n^{\alpha\beta}) \left[\bar{u}_{3,\beta} - \bar{h}(\bar{\psi}_{3,\beta} - ''\psi_{3,\beta}) \right] + \\
& + (''k^{\alpha\beta} + \bar{h}\,''m^{\alpha\beta})\,''\psi_{3,\beta} + ''q^\alpha\,\bar{h}\,(1 + ''\psi_3) \Big\}\Big|_\alpha + \\
& + (''t^\alpha|_\alpha - ''n^{33})\,(1 + ''\psi_3) - ''q^\alpha \left[\bar{u}_{3,\alpha} - \bar{h}(\bar{\psi}_{3,\alpha} - \right. \\
& \left. - ''\psi_{3,\alpha}) \right] + ''g^3 + ''d^3 = 0 \quad .
\end{aligned} \tag{151h}$$

It is easy to see that our results can be reduced to those obtained by Ebcioglu (11), and shown by Ebcioglu to contain the earlier works of Eringen (21) and Yu (22).

For homogeneous shells, we simply drop the upper and lower facings of the composite shell and retain only the core layer. Then, all terms including quantities denoted by single and double primes and the last four equations of Equations (105) should be suppressed. We have

$$\begin{aligned}
\delta\,\bar{u}_\delta : \quad & \Big\{ \bar{n}^{\alpha\beta}(\delta^\delta_\beta + \bar{u}^\delta\|_\beta - b^\delta_\beta\,\bar{u}_3) + \bar{m}^{\alpha\beta}(\bar{\psi}^\delta\|_\beta - b^\delta_\beta\,\bar{\psi}_3) - \\
& - \bar{m}^{\alpha\beta} b^\delta_\beta + \bar{q}^\alpha\,\bar{\psi}^\delta \Big\}\Big\|_\alpha - \Big\{ \bar{n}^{\alpha\beta}(\bar{u}_{3,\beta} + b^\lambda_\beta\,\bar{u}_\lambda) + \\
& + \bar{m}^{\alpha\beta}(\bar{\psi}_{3,\beta} + b^\lambda_\beta\,\bar{\psi}_\lambda) + \bar{q}^\alpha(1 + \bar{\psi}_3) \Big\}\, b^\delta_\alpha + \\
& + \bar{p}^\delta - \bar{f}^\delta = 0 \quad ,
\end{aligned} \tag{152a}$$

$$\delta\bar{u}_3: \quad \left\{ \bar{n}^{\alpha\beta}(\bar{u}_{3,\beta} + b^{\delta}_{\beta}\bar{u}_{\delta}) + \bar{m}^{\alpha\beta}(\bar{\Psi}_{3,\beta} + b^{\delta}_{\beta}\bar{\Psi}_{\delta}) + \right.$$

$$\left. + \bar{q}^{\alpha}(1 + \bar{\Psi}_3) \right\}\|_{\alpha} + \left\{ \bar{n}^{\beta\delta}(\delta^{\alpha}_{\delta} + \bar{u}^{\alpha}\|_{\delta} - b^{\alpha}_{\delta}\bar{u}_3) + \right.$$

$$\left. + \bar{m}^{\beta\delta}(\bar{\Psi}^{\alpha}\|_{\delta} - b^{\alpha}_{\delta}\bar{\Psi}_3) - \bar{m}^{\beta\delta} b^{\alpha}_{\delta} + \bar{q}^{\beta}\bar{\Psi}^{\alpha} \right\} b_{\alpha\beta} +$$

$$+ \bar{p}^3 - \bar{f}^3 = 0 \quad , \tag{152b}$$

$$\delta\bar{\Psi}_{\delta}: \quad \left\{ \bar{m}^{\alpha\beta}(\delta^{\delta}_{\beta} + \bar{u}^{\delta}\|_{\beta} - b^{\delta}_{\beta}\bar{u}_3) - \bar{k}^{\alpha\beta} b^{\delta}_{\beta} + \bar{k}^{\alpha\beta}(\bar{\Psi}^{\delta}\|_{\beta} - \right.$$

$$\left. - b^{\delta}_{\beta}\bar{\Psi}_3) \right\}\|_{\alpha} - \left\{ \bar{m}^{\alpha\beta}(\bar{u}_{3,\beta} + b^{\lambda}_{\beta}\bar{u}_{\lambda}) + \bar{k}^{\alpha\beta}(\bar{\Psi}_{3,\beta} + \right.$$

$$\left. + b^{\lambda}_{\beta}\bar{\Psi}_{\lambda}) \right\} b^{\delta}_{\alpha} + (\bar{t}^{\alpha}\|_{\alpha} - \bar{n}^{33})\bar{\Psi}^{\delta} - \bar{q}^{\alpha}(\delta^{\delta}_{\alpha} +$$

$$+ \bar{u}^{\delta}\|_{\alpha} - b^{\delta}_{\alpha}\bar{u}_3) + \bar{c}^{\delta} - \bar{m}^{\delta} = 0 \quad , \tag{152c}$$

$$\delta\bar{\Psi}_3: \quad \left\{ \bar{m}^{\alpha\beta}(\bar{u}_{3,\beta} + b^{\delta}_{\beta}\bar{u}_{\delta}) + \bar{k}^{\alpha\beta}(\bar{\Psi}_{3,\beta} + b^{\delta}_{\beta}\bar{\Psi}_{\delta}) \right\}\|_{\alpha} +$$

$$+ \left\{ \bar{m}^{\beta\delta}(\delta^{\alpha}_{\delta} + \bar{u}^{\alpha}\|_{\delta} - b^{\alpha}_{\delta}\bar{u}_3) + \bar{k}^{\beta\delta}(\bar{\Psi}^{\alpha}\|_{\delta} - b^{\alpha}_{\delta}\bar{\Psi}_3) - \right.$$

$$\left. - \bar{k}^{\beta\delta} b^{\alpha}_{\delta} \right\} b_{\alpha\beta} - \bar{q}^{\alpha}(\bar{u}_{3,\alpha} + b^{\beta}_{\alpha}\bar{u}_{\beta}) +$$

$$+ (\bar{t}^{\alpha}\|_{\alpha} - \bar{n}^{33})(1 + \bar{\Psi}_3) + \bar{c}^3 - \bar{m}^3 = 0 \quad . \tag{152d}$$

This is in agreement with the results of a nonlinear theory of elastic shells obtained by Habip (13), (14). In the absence of curvature effects, these also reduce to those of Habip (13) for plates in the reference state.

In the partially nonlinear case, our Equations (135) agree with the results of Sanders (24), given for the small strain approximation, if we drop the effects of the facings and adopt Kirchhoff's hypothesis for the resulting homogeneous shell.

By setting $'\Psi_3 = ''\Psi_3 = \bar{\Psi}_3 = 0$, and noting that

$$'\hat{n}^{\alpha\beta} = 'n^{\alpha\beta} - b^\beta_\delta\, 'm^{\alpha\delta} \quad , \quad '\hat{m}^{\alpha\beta} = 'm^{\alpha\beta} - b^\beta_\delta\, 'k^{\alpha\delta} \quad , \quad \hat{n}^{\alpha\beta} = '\hat{n}^{\alpha\beta} + \hat{\bar{n}}^{\alpha\beta} + ''\hat{n}^{\alpha\beta} \quad ,$$

Equations (143) lead to

$$\delta \bar{u}_\delta : \qquad \hat{n}^{\delta\alpha}\|_\alpha - q^\alpha b^\delta_\alpha + p^\delta - f^\delta = 0 \quad , \tag{153a}$$

$$\delta \bar{u}_3 : \qquad (n^{\alpha\beta} \bar{u}_{3,\beta} + q^\alpha)\|_\alpha + \hat{n}^{\alpha\beta} b_{\alpha\beta} + p^3 - f^3 = 0 \quad , \tag{153b}$$

$$\delta \bar{\Psi}_\delta : \qquad [\, \hat{\bar{m}}^{\alpha\delta} + \bar{h}\,('\hat{n}^{\alpha\delta} - ''\hat{n}^{\alpha\delta})\,]\|_\alpha - \bar{h}\,('q^\alpha - ''q^\alpha)\, b^\delta_\alpha - \bar{q}^\delta + C^\delta - m^\delta = 0 \quad , \tag{153c}$$

$$\delta '\Psi_\delta : \qquad ('\hat{m}^{\alpha\beta} - \bar{h}\, '\hat{n}^{\alpha\beta})\|_\alpha + 'q^\alpha \bar{h}\, b^\delta_\alpha - 'q^\delta + 'g^\delta - 'd^\delta = 0 \quad , \tag{153d}$$

$$\delta ''\Psi_\delta : \qquad (''\hat{m}^{\alpha\beta} + \bar{h}\, ''\hat{n}^{\alpha\beta})\|_\alpha - ''q^\alpha \bar{h}\, b^\delta_\alpha - ''q^\delta + ''g^\delta + ''d^\delta = 0 \quad , \tag{153e}$$

These are identical with Ebcioglu's equations (12).

For comparison with the well-known works of Reissner (6), (7), we transform Equations (147) into their conventional form, in terms of physical components, as shown in the Appendix. In order to show that these previous results are contained in (147) we use the following notation.

$$\theta_1 = x \quad , \quad \theta_2 = y \quad , \quad \sqrt{a_{11}} = \sqrt{a_{22}} = 1 \quad , \quad R_1 = R_2 = \infty$$

$$\hat{N}_{11} = N_{11} = N_x \quad , \quad \hat{N}_{12} = N_{12} = N_{xy} \quad , \quad \hat{N}_{22} = N_{22} = N_y \quad ,$$

$$Q_1 = Q_x \quad , \quad Q_2 = Q_y \quad , \quad \hat{M}_{11} = M_{11} = M_x \quad ,$$

$$\hat{M}_{12} = M_{12} = M_{xy} \quad , \quad \hat{M}_{22} = M_{22} = M_y \quad , \quad P_1 = P_2 = C_1 = C_2 = 0 \quad ,$$

and also omit the effects of thermal gradients and acceleration resultants. Equations (A8) become

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad , \tag{154a}$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \quad , \tag{154b}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \quad , \tag{154c}$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0 \quad , \tag{154d}$$

$$\begin{aligned}\frac{\partial Q_x}{\partial x} &+ \frac{\partial Q_y}{\partial y} + N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + \\ &+ N_y \frac{\partial^2 w}{\partial y^2} + M_x \frac{\partial^2 e}{\partial x^2} + 2 M_{xy} \frac{\partial^2 e}{\partial x \partial y} + \\ &+ M_y \frac{\partial^2 e}{\partial y^2} + P = 0 \quad ,\end{aligned} \tag{154e}$$

$$\begin{aligned}\frac{1}{2\bar{h}} &\left(M_x \frac{\partial^2 w}{\partial x^2} + 2 M_{xy} \frac{\partial^2 w}{\partial x \partial y} + M_y \frac{\partial^2 w}{\partial y^2} \right) + \\ &+ \frac{\bar{h}}{2} \left(N_x \frac{\partial^2 e}{\partial x^2} + 2 N_{xy} \frac{\partial^2 e}{\partial x \partial y} + N_y \frac{\partial^2 e}{\partial y^2} \right) + \\ &- \sigma_{3m} (1 + e) + \underset{\sim}{S} = 0 \quad .\end{aligned} \tag{154f}$$

These are the quations given by Reissner (6), (7).

Completely linearizing Equations (A8) and neglecting the acceleration resultants, we obtain

$$\begin{aligned}\frac{\partial}{\partial \theta_1} &\left(\sqrt{a_{22}}\, \hat{N}_{11} \right) + \frac{\partial}{\partial \theta_2} \left(\sqrt{a_{11}}\, \hat{N}_{21} \right) + \frac{\partial \sqrt{a_{11}}}{\partial \theta_2} \hat{N}_{12} - \\ &- \frac{\partial \sqrt{a_{22}}}{\partial \theta_1} \hat{N}_{22} + \sqrt{a} \left(\frac{Q_1}{R_1} + P_1 \right) = 0 \quad ,\end{aligned} \tag{155a}$$

$$\begin{aligned}\frac{\partial}{\partial \theta_2} &\left(\sqrt{a_{11}}\, \hat{N}_{22} \right) + \frac{\partial}{\partial \theta_1} \left(\sqrt{a_{22}}\, \hat{N}_{12} \right) + \frac{\partial \sqrt{a_{22}}}{\partial \theta_1} \hat{N}_{21} - \\ &- \frac{\partial \sqrt{a_{11}}}{\partial \theta_2} \hat{N}_{11} + \sqrt{a} \left(\frac{Q_2}{R_2} + P_2 \right) = 0 \quad ,\end{aligned} \tag{155b}$$

$$\frac{\partial}{\partial\theta_1}(\sqrt{a_{22}}\,Q_1) + \frac{\partial}{\partial\theta_2}(\sqrt{a_{11}}\,Q_2) - \sqrt{a}\left(\frac{\hat{N}_{11}}{R_1} + \right.$$

$$\left. + \frac{\hat{N}_{22}}{R_2}\right) + \sqrt{a}\,P = 0 \quad , \tag{155c}$$

$$\frac{\partial}{\partial\theta_1}(\sqrt{a_{22}}\,\hat{M}_{11}) + \frac{\partial}{\partial\theta_2}(\sqrt{a_{11}}\,\hat{M}_{21}) + \frac{\partial\sqrt{a_{11}}}{\partial\theta_2}\,\hat{M}_{12} -$$

$$- \frac{\partial\sqrt{a_{22}}}{\partial\theta_1}\,\hat{M}_{22} + \sqrt{a}\,(C_1 - Q_1) = 0 \quad , \tag{155d}$$

$$\frac{\partial}{\partial\theta_2}(\sqrt{a_{11}}\,\hat{M}_{22}) + \frac{\partial}{\partial\theta_1}(\sqrt{a_{22}}\,\hat{M}_{12}) + \frac{\partial\sqrt{a_{22}}}{\partial\theta_1}\,\hat{M}_{21} -$$

$$- \frac{\partial\sqrt{a_{11}}}{\partial\theta_2}\,\hat{M}_{11} + \sqrt{a}\,(C_2 - Q_2) = 0 \quad , \tag{155e}$$

$$\sigma_{3m} + \frac{1}{2\bar{h}}\left(\frac{\hat{M}_{11}}{R_1} + \frac{\hat{M}_{22}}{R_2}\right) = 0 \quad , \tag{155f}$$

These are the equations obtained by Reissner (5) for the small bending and stretching of sandwich-type shells.

The above comparisons reveal that the modified Hellinger-Reissner theorem is particularly suitable for the consistent derivation of intermediate theories. However, during the course of this derivation, we found that the order of the variation and "shifting" is important. If we execute the indicated variation in Equation (86) before shifting the components of displacement, i.e., before we employ the normal coordinates in the three-dimensional body, some inconsistency will appear between strain-displacement relations and the equations of motion, forcing us to drop certain terms in the strain-displacement relations in order to make them compatible with the equations of motion (14). This

arises only in partially nonlinear theories. For the general non-linear case, the order of variation and "shifting" is immaterial.

APPENDIX

For the purpose of comparison, we now proceed to record the physical components of the equations of motion (147) for the nonlinear membrane theory of sandwich shells with a weak core and identical facings, i.e., $'h = ''h = t$. The material of the composite shell is isotropic, and the coordinates are in lines of curvature. The body forces are neglected.

Following Ebcioglu's procedure (12), assuming the membrane and transverse shear stresses to be uniform across the thickness of the facings and the core, respectively, and with the transverse normal stress given as

$$\bar{S}^{33} = \frac{\sigma_{3+} + \sigma_{3-}}{2} + \frac{\theta_3}{2\bar{h}}(\sigma_{3+} - \sigma_{3-}) \quad , \tag{A1}$$

we obtain the stress and couple resultants in terms of physical components as follows

$$'\hat{n}_{(11)} = \int_{\bar{h}}^{\bar{h}+2t} {'S}_{(11)} \left(1 + \frac{\theta_3}{R_2}\right) d\theta_3 = {'N}_{11}\left(1 + \frac{\bar{h}+t}{R_2}\right) ,$$

$$'\hat{n}_{(22)} = \int_{\bar{h}}^{\bar{h}+2t} {'S}_{(22)} \left(1 + \frac{\theta_3}{R_1}\right) d\theta_3 = {'N}_{22}\left(1 + \frac{\bar{h}+t}{R_1}\right) ,$$

$$'n_{(11)} = \int_{\bar{h}}^{\bar{h}+2t} {'S}_{(11)}\, d\theta_3 = {'N}_{11} ,$$

$$'n_{(22)} = \int_{\bar{h}}^{\bar{h}+2t} {'S}_{(22)}\, d\theta_3 = {'N}_{22} ,$$

$$'\hat{n}_{(12)} = \int_{\bar{h}}^{\bar{h}+2t} {'S}_{(12)} \left(1+\frac{\theta_3}{R_2}\right) d\theta_3 = {'N}_{12} \left(1+\frac{\bar{h}+t}{R_2}\right) ,$$

$$'\hat{n}_{(21)} = \int_{\bar{h}}^{\bar{h}+2t} {'S}_{(21)} \left(1+\frac{\theta_3}{R_1}\right) d\theta_3 = {'N}_{21} \left(1+\frac{\bar{h}+t}{R_1}\right) ,$$

$$'n_{(12)} = {'n}_{(21)} = {'N}_{12} = {'N}_{21} ,$$

$$''\hat{n}_{(11)} = \int_{-\bar{h}-2t}^{-\bar{h}} {''S}_{(11)} \left(1+\frac{\theta_3}{R_2}\right) d\theta_3 = {''N}_{11} \left(1-\frac{\bar{h}+t}{R_2}\right) ,$$

$$''\hat{n}_{(22)} = \int_{-\bar{h}-2t}^{-\bar{h}} {''S}_{(22)} \left(1+\frac{\theta_3}{R_1}\right) d\theta_3 = {''N}_{22} \left(1-\frac{\bar{h}+t}{R_1}\right) ,$$

$$''n_{(11)} = {''N}_{11} , \qquad {''n}_{(22)} = {''N}_{22} ,$$

$$''\hat{n}_{(12)} = \int_{-\bar{h}-2t}^{-\bar{h}} {''S}_{(12)} \left(1+\frac{\theta_3}{R_2}\right) d\theta_3 = {''N}_{12} \left(1-\frac{\bar{h}+t}{R_2}\right) ,$$

$$''\hat{n}_{(21)} = \int_{-\bar{h}-2t}^{-\bar{h}} {''S}_{(21)} \left(1+\frac{\theta_3}{R_1}\right) d\theta_3 = {''N}_{21} \left(1-\frac{\bar{h}+t}{R_1}\right) , \qquad \text{(A2)}$$

$$''n_{(12)} = {''n}_{(21)} = {''N}_{12} = {''N}_{21} ,$$

$$\bar{q}_{(1)} = \int_{-\bar{h}}^{\bar{h}} \bar{S}_{(13)} \, d\theta_3 = Q_1 ,$$

$$\bar{q}_{(2)} = \int_{-\bar{h}}^{\bar{h}} \bar{S}_{(23)} \, d\theta_3 = Q_2 ,$$

$$\bar{t}_{(1)} = \int_{-\bar{h}}^{\bar{h}} \bar{S}_{(13)} \, \theta_3 \, d\theta_3 = 0 ,$$

$$\bar{t}_{(2)} = \int_{-\bar{h}}^{\bar{h}} \bar{S}_{(23)} \, \theta_3 \, d\theta_3 = 0 ,$$

$$\bar{n}^{33} = \int_{-\bar{h}}^{\bar{h}} \bar{S}_{(33)}\, d\theta_3 = \frac{\sigma_{3+} + \sigma_{3-}}{2} \cdot 2\bar{h}$$

$$= 2\bar{h}\,\sigma_{3m}$$

where

$$\begin{aligned}
&{}'N_{11} = {}'S_{(11)}\,2t \quad , \qquad {}'N_{22} = {}'S_{(22)}\,2t \quad , \\
&{}'N_{12} = {}'S_{(12)}\,2t = {}'N_{21} = {}'S_{(21)}\,2t \quad , \\
&{}''N_{11} = {}''S_{(11)}\,2t \quad , \qquad {}''N_{22} = {}''S_{(22)}\,2t \quad , \\
&{}''N_{12} = {}''S_{(12)}\,2t = {}''N_{21} = {}''S_{(21)}\,2t \quad , \\
&\sigma_{3m} = \frac{1}{2}\,(\sigma_{3+} + \sigma_{3-})
\end{aligned} \tag{A3}$$

and σ_{3m} represents the value of the effective transverse normal stress.

From Equations (A2), we can deduce the following definitions

$$\hat{n}_{(11)} = {}'\hat{n}_{(11)} + {}''\hat{n}_{(11)} =$$

$$= \left(1 + \frac{\bar{h}+t}{R_2}\right){}'N_{11} + \left(1 - \frac{\bar{h}+t}{R_2}\right){}''N_{11} = \hat{N}_{11} \quad ,$$

$$n_{(11)} = {}'n_{(11)} + {}''n_{(11)} = {}'N_{11} + {}''N_{11} = N_{11} \quad ,$$

$$\hat{n}_{(22)} = {}'\hat{n}_{(22)} + {}''\hat{n}_{(22)} =$$

$$= \left(1 + \frac{\bar{h}+t}{R_1}\right){}'N_{22} + \left(1 - \frac{\bar{h}+t}{R_1}\right){}''N_{22} \quad ,$$

$$n_{(22)} = {}'n_{(22)} + {}''n_{(22)} = {}'N_{22} + {}''N_{22} = N_{22} \quad ,$$

$$\hat{n}_{(12)} = {}'\hat{n}_{(12)} + {}''\hat{n}_{(12)} =$$

$$= \left(1 + \frac{\bar{h}+t}{R_2}\right){}'N_{12} + \left(1 - \frac{\bar{h}+t}{R_2}\right){}''N_{12} = \hat{N}_{12} \quad ,$$

$$n_{(12)} = {}'n_{(12)} + {}''n_{(12)} = {}'N_{12} + {}''N_{12} = N_{12} = N_{21} \quad ,$$

$$\hat{n}_{(21)} = {}'\hat{n}_{(21)} + {}''\hat{n}_{(21)} =$$

$$= \left(1 + \frac{\bar{h}+t}{R_1}\right){}'N_{21} + \left(1 - \frac{\bar{h}+t}{R_1}\right){}''N_{21} = \hat{N}_{21} \quad ,$$

$$\bar{h}\left({}'\hat{n}_{(11)} - {}''\hat{n}_{(11)}\right) = \bar{h}\left[\left(1 + \frac{\bar{h}+t}{R_1}\right){}'N_{21} - \left(1 - \frac{\bar{h}+t}{R_1}\right){}''N_{11}\right]$$

$$= \hat{M}_{11} \quad , \tag{A4}$$

$$\bar{h}\left({}'n_{(11)} - {}''n_{(11)}\right) = \bar{h}\left({}'N_{11} - {}''N_{11}\right) = M_{11} \quad ,$$

$$\bar{h}\left({}'\hat{n}_{(22)} - {}''\hat{n}_{(22)}\right) = \bar{h}\left[\left(1 + \frac{\bar{h}+t}{R_1}\right){}'N_{22} - \left(1 - \frac{\bar{h}+t}{R_1}\right)\right] = \hat{M}_{22} \quad ,$$

$$\bar{h}\left({}'n_{(22)} - {}''n_{(22)}\right) = \bar{h}\left({}'N_{22} - {}''N_{22}\right) = M_{22} \quad ,$$

$$\bar{h}\left({}'\hat{n}_{(12)} - {}''\hat{n}_{(12)}\right) = \bar{h}\left[\left(1 + \frac{\bar{h}+t}{R_2}\right){}'N_{12} - \left(1 - \frac{\bar{h}+t}{R_2}\right){}''N_{12}\right]$$

$$= \hat{M}_{12} \quad ,$$

$$\bar{h}\left({}'n_{(12)} - {}''n_{(12)}\right) = \bar{h}\left({}'N_{12} - {}''N_{12}\right) = M_{12} = M_{21} \quad ,$$

$$\bar{h}\left({}'\hat{n}_{(21)} - {}''\hat{n}_{(21)}\right) = \bar{h}\left[\left(1 + \frac{\bar{h}+t}{R_1}\right){}'N_{21} -\right.$$

$$\left. - \left(1 - \frac{\bar{h}+t}{R_1}\right){}''N_{21}\right] = \hat{M}_{21} \quad .$$

In the same way, the following effective external force and moment intensities are obtained

$$
\begin{aligned}
p_{(\alpha)} &= \left(1+\frac{\bar{h}+t}{R_2}\right)\left(1+\frac{\bar{h}+t}{R_1}\right)\tilde{p}_{+(\alpha)} + \\
&\quad + \left(1-\frac{\bar{h}+t}{R_2}\right)\left(1-\frac{\bar{h}+t}{R_1}\right)\tilde{p}_{-(\alpha)} = P_{\alpha} \quad , \\
p_{(3)} &= \left(1+\frac{\bar{h}+t}{R_1}\right)\left(1+\frac{\bar{h}+t}{R_1}\right)\tilde{p}_{+(3)} + \\
&\quad + \left(1-\frac{\bar{h}+t}{R_2}\right)\left(1-\frac{\bar{h}+t}{R_1}\right)\tilde{p}_{-(3)} = P \quad , \\
C_{(\alpha)} &= \bar{h}\left[\left(1+\frac{\bar{h}+t}{R_2}\right)\left(1+\frac{\bar{h}+t}{R_1}\right)\tilde{p}_{+(\alpha)} - \right. \\
&\quad \left. - \left(1-\frac{\bar{h}+t}{R_2}\right)\left(1-\frac{\bar{h}+t}{R_1}\right)\tilde{p}_{-(\alpha)}\right] = C_{\alpha} \quad , \\
C_{(3)} &= \bar{h}\left[\left(1+\frac{\bar{h}+t}{R_2}\right)\left(1+\frac{\bar{h}+t}{R_1}\right)\tilde{p}_{+(3)} - \right. \\
&\quad \left. - \left(1-\frac{\bar{h}+t}{R_2}\right)\left(1-\frac{\bar{h}+t}{R_1}\right)\tilde{p}_{-(3)}\right] = C_3 = 2\bar{h}\,\underset{\sim}{S}
\end{aligned}
\qquad \text{(A5)}
$$

where $\tilde{p}_{+(i)}$ and $\tilde{p}_{-(i)}$ are the prescribed loading at the upper and lower facings, respectively, and

$$
\underset{\sim}{S} = \frac{1}{2}\left[\left(1+\frac{\bar{h}+t}{R_1}\right)\left(1+\frac{\bar{h}+t}{R_2}\right)\tilde{p}_{+(3)} - \left(1-\frac{\bar{h}+t}{R_2}\right)\left(1-\frac{\bar{h}+t}{R_1}\right)\tilde{p}_{-(3)}\right] \quad . \qquad \text{(A6)}
$$

If we let

$$
\begin{aligned}
u &= \bar{u}_{(1)} \quad , & v &= \bar{u}_{(2)} \quad , \\
w &= \bar{u}_{(3)} \quad , & e &= \bar{\Psi}_{(3)} \quad ,
\end{aligned}
\qquad \text{(A7)}
$$

where e represents the effective transverse normal strain for the composite shell, the equations of motion (147) then become

$$
\frac{\partial}{\partial \theta_1}\left(\sqrt{a_{11}}\,\hat{N}_{11}\right) + \frac{\partial}{\partial \theta_2}\left(\sqrt{a_{11}}\,\hat{N}_{21}\right) + \frac{\partial \sqrt{a_{11}}}{\partial \theta_2}\hat{N}_{12} - \frac{\partial \sqrt{a_{22}}}{\partial \theta_1}\hat{N}_{22} +
$$

$$+\sqrt{a}\left(\frac{Q_1}{R_1}+P_1-\mathcal{F}_1\right)=0 \quad , \tag{A8a}$$

$$\frac{\partial}{\partial\theta_2}(\sqrt{a_{11}}\,\hat{N}_{22})+\frac{\partial}{\partial\theta_1}(\sqrt{a_{22}}\,\hat{N}_{12})+\frac{\partial\sqrt{a_{22}}}{\partial\theta_1}\hat{N}_{21}-\frac{\partial\sqrt{a_{11}}}{\partial\theta_2}\hat{N}_{11}+$$

$$+\sqrt{a}\left(\frac{Q_2}{R_2}+P_2-\mathcal{F}_2\right)=0 \quad , \tag{A8b}$$

$$\frac{\partial}{\partial\theta_1}\left(\sqrt{a_{22}}\,\frac{N_{11}}{\sqrt{a_{11}}}\frac{\partial w}{\partial\theta_1}+N_{12}\frac{\partial w}{\partial\theta_2}\right)+\frac{\partial}{\partial\theta_2}\left(N_{21}\frac{\partial w}{\partial\theta_1}\right)+$$

$$+\frac{\partial}{\partial\theta_2}\left(\sqrt{a_{11}}\,\frac{N_{22}}{\sqrt{a_{22}}}\frac{\partial w}{\partial\theta_2}\right)+\frac{\partial}{\partial\theta_1}\left(\sqrt{a_{22}}\,\frac{M_{11}}{\sqrt{a_{11}}}\frac{\partial e}{\partial\theta_1}+M_{12}\frac{\partial e}{\partial\theta_2}\right)+$$

$$+\frac{\partial}{\partial\theta_2}\left(M_{21}\frac{\partial e}{\partial\theta_1}+\sqrt{a_{11}}\,\frac{M_{22}}{\sqrt{a_{22}}}\frac{\partial e}{\partial\theta_2}\right)+\frac{\partial}{\partial\theta_1}(\sqrt{a_{22}}\,Q_1)+$$

$$+\frac{\partial}{\partial\theta_2}(\sqrt{a_{11}}\,Q_2)+\sqrt{a_{22}}\,Q_1\frac{\partial e}{\partial\theta_1}+\sqrt{a_{11}}\,Q_2\frac{\partial e}{\partial\theta_2}-$$

$$-\sqrt{a}\left(\frac{\hat{N}_{11}}{R_1}+\frac{\hat{N}_{22}}{R_2}-P+\mathcal{F}\right) \tag{A8c}$$

$$\frac{\partial}{\partial\theta_1}(\sqrt{a_{22}}\,\hat{M}_{11})+\frac{\partial}{\partial\theta_2}(\sqrt{a_{11}}\,\hat{M}_{21})+\frac{\partial\sqrt{a_{11}}}{\partial\theta_2}\hat{M}_{12}-$$

$$-\frac{\partial\sqrt{a_{22}}}{\partial\theta_1}\hat{M}_{22}-\sqrt{a}\,(Q_1-C_1+\mathcal{M}_1)=0 \quad , \tag{A8d}$$

$$\frac{\partial}{\partial\theta_2}(\sqrt{a_{11}}\,\hat{M}_{22})+\frac{\partial}{\partial\theta_1}(\sqrt{a_{22}}\,\hat{M}_{12})+\frac{\partial\sqrt{a_{22}}}{\partial\theta_1}\hat{M}_{21}-$$

$$-\frac{\partial\sqrt{a_{11}}}{\partial\theta_2}\hat{M}_{12}-\sqrt{a}\,(Q_2-C_2+\mathcal{M}_2)=0 \quad , \tag{A8e}$$

$$\frac{\partial}{\partial\theta_1}\left(\sqrt{a_{22}}\,\frac{M_{11}}{\sqrt{a_{11}}}\frac{\partial w}{\partial\theta_1}+M_{12}\frac{\partial w}{\partial\theta_2}\right)+\frac{\partial}{\partial\theta_2}\left(M_{21}\frac{\partial w}{\partial\theta_1}+\right.$$

$$\left.+\sqrt{a_{11}}\,\frac{M_{22}}{\sqrt{a_{22}}}\frac{\partial w}{\partial\theta_2}\right)+\bar{h}^2\left[\frac{\partial}{\partial\theta_1}\left(\sqrt{a_{22}}\,\frac{N_{11}}{\sqrt{a_{11}}}\frac{\partial e}{\partial\theta_1}+\right.\right.$$

$$\left.\left.+N_{12}\frac{\partial e}{\partial\theta_2}\right)+\frac{\partial}{\partial\theta_2}\left(N_{21}\frac{\partial e}{\partial\theta_1}+\sqrt{a_{11}}\,\frac{N_{22}}{\sqrt{a_{22}}}\frac{\partial e}{\partial\theta_2}\right)\right]-$$

$$-\sqrt{a_{22}}\,Q_1\frac{\partial w}{\partial\theta_1}-\sqrt{a_{11}}\,Q_2\frac{\partial w}{\partial\theta_2}-\sqrt{a}\left[\frac{\hat{M}_{11}}{R_1}+\right.$$

$$\left.+\frac{\hat{M}_{22}}{R_2}+2\bar{h}\,\sigma_{3m}(1+e)-2\bar{h}\,\underset{\sim}{S}+\mathcal{M}\right]=0 \quad , \tag{A8f}$$

where $\mathcal{F}_\alpha$, $\mathcal{F}_3$, $\mathcal{M}_\alpha$, $\mathcal{M}_3$ represent the physical components of the inertial forces and moment resultants.

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BIOGRAPHICAL SKETCH

Ju-Chin Huang was born on July 2, 1934, at Canton, China. In August, 1953, he was graduated from Chee Yung High School, Cholon, South Viet Nam. In July, 1958, he received the degree of Bachelor of Science in Civil Engineering from Cheng Kung University, Tainan, Taiwan, Republic of China. From August, 1958, until December, 1961, he served as a teaching assistant in his Alma Mater.

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